Astrophysical Dynamics, VT 2010

Gas Dynamics:
Basic Equations, Waves and Shocks

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Gas Dynamics: Basic Equations, Waves and Shocks

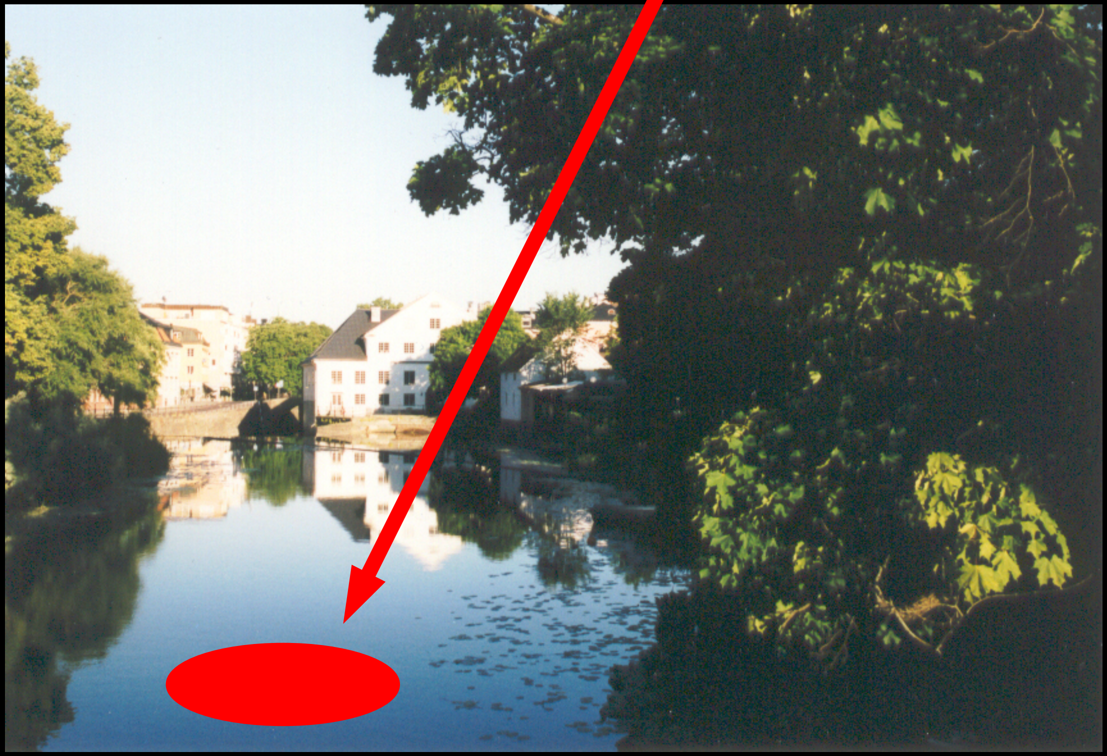
1. Basic Concepts of Fluid Mechanics

Fluid Dynamics: Basic Concepts

consider a typical fluid ...

Fluid Dynamics: Basic Concepts

... and define a fluid element



Fluid Dynamics: Basic Concepts

Definition of fluid element:

A region over which we can define local variables (density, temperature, etc.) The size of this region (length scale l_{region}) is assumed to be such that it is

(i) small enough that we can ignore systematic variations across it for any variable q we are interested in:

$$l_{\text{region}} \ll l_{\text{scale}} \sim q / |\nabla q|$$

(ii) large enough to contain sufficient particles to ignore fluctuations due to the finite number of particles (discreteness noise):

$$n\, l^3_{\text{region}} \gg 1$$

In addition, collisional fluids must satisfy:

(iii) $l_{\text{region}} \gg$ mean free path of particles

Mean Free Path - Examples

Mean Free Path

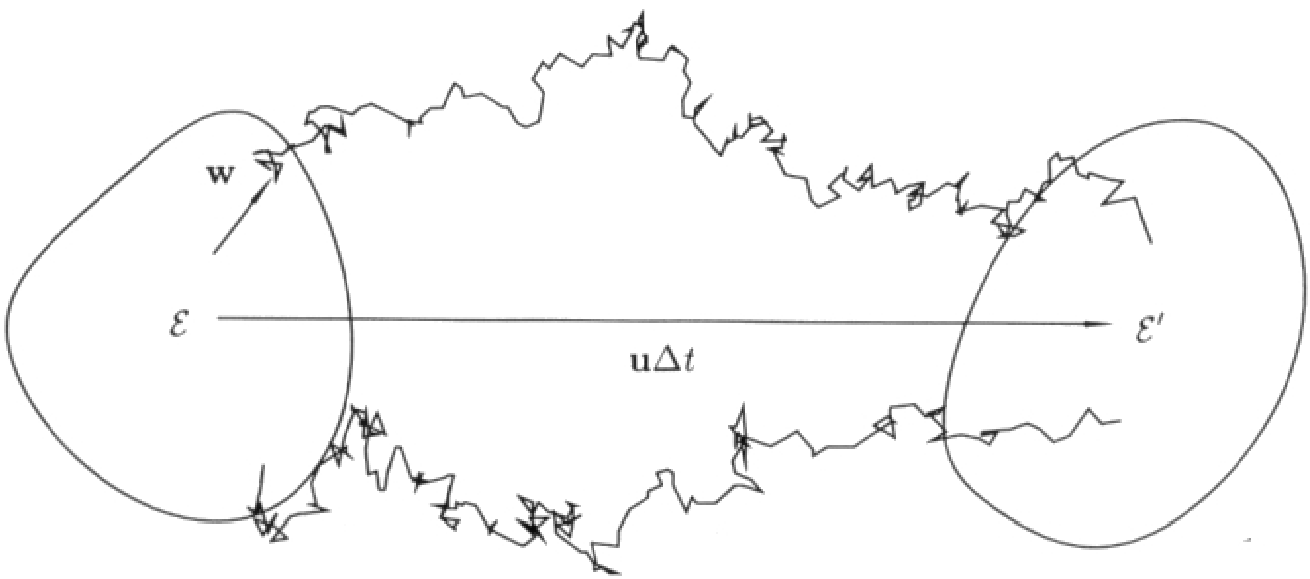
- depends on number density and cross section
- elastic scattering cross section of neutral atoms: 10^{-15} cm^2

Examples

gas in	density [cm^{-3}]	MFP [cm]	size [cm]
typical room	10^{19}	10^{-4}	10^{3}
HI region (ISM)	10	10^{14}	10^{19}

Fluid Dynamics: Basic Concepts

- frequent collisions, small mean free path compared to the characteristic length scales of the system → coherent motion of particles
- definition of fluid element: mean flow velocity (bulk velocity)
- particle velocity = mean flow velocity + random velocity component



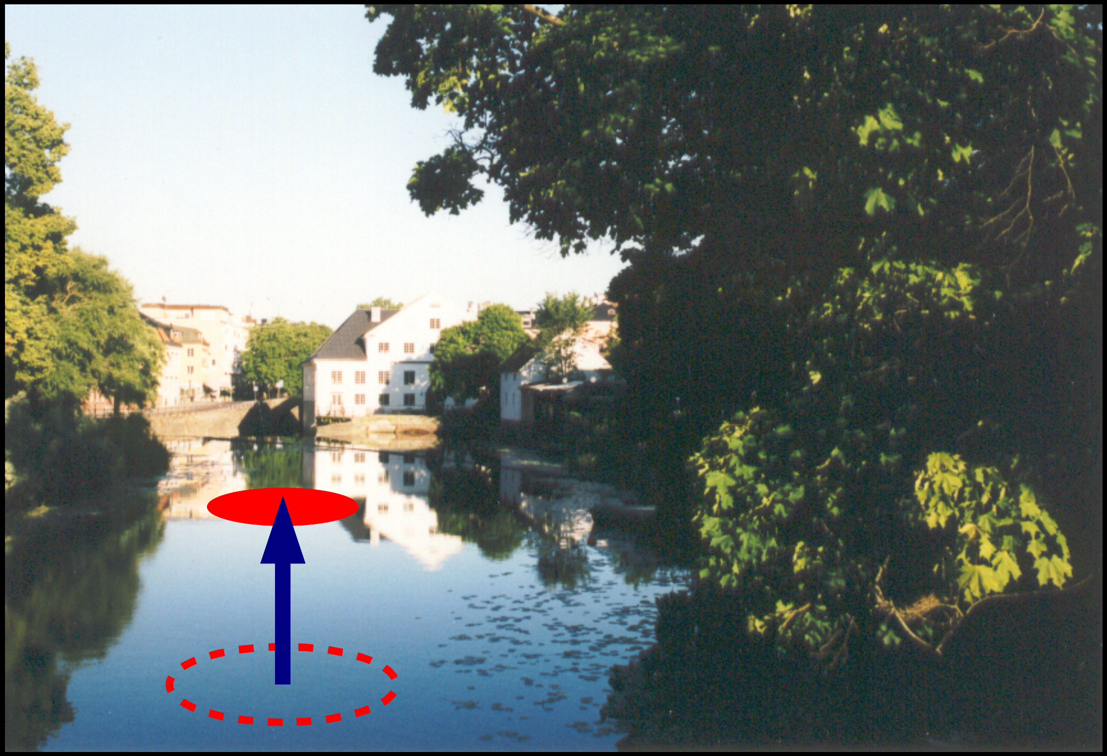
random-walk trajectory

Fluid Dynamics: Basic Concepts

coherent motion of particles

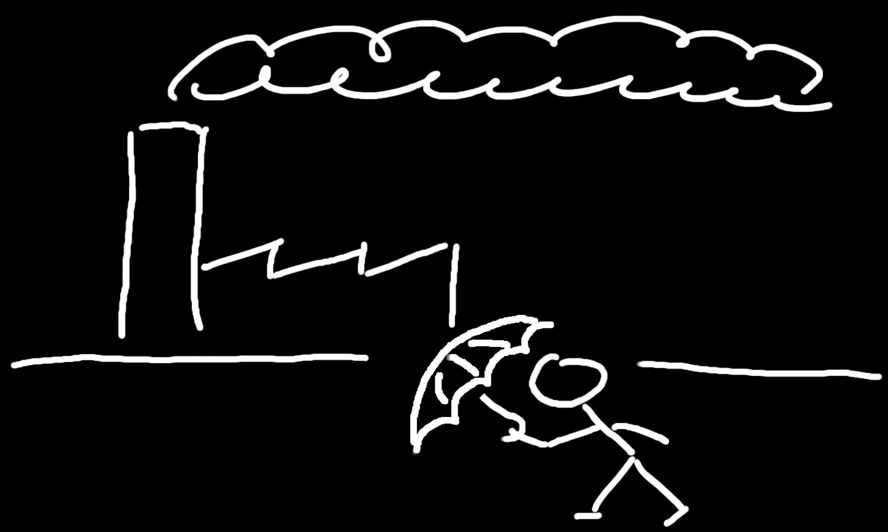
Fluid Dynamics: Basic Concepts

coherent motion of particles

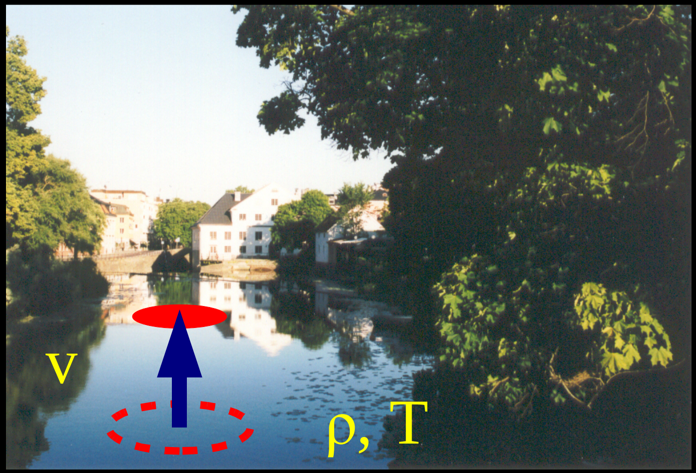


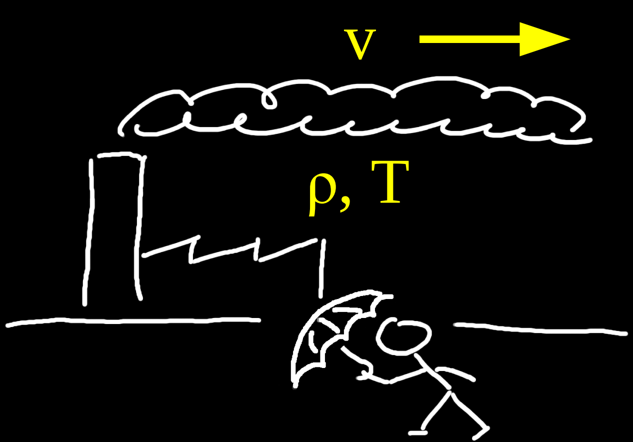
Fluid Dynamics: Basic Concepts

coherent motion of particles



Fluid Dynamics: Basic Concepts





- The fluid is described by local macroscopic variables (e.g., density, temperature, bulk velocity)
- The dynamics of the fluid is governed by internal forces (e.g. pressure gradients) and external forces (e.g. gravity)
- The changes of macroscopic properties in a fluid element are described by conservation laws

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Gas Dynamics: Basic Equations, Waves and Shocks

2. Fluid Dynamics: Conservation Laws

Fluid Dynamics: The Conservation Laws

Mass conservation:

Fluid Dynamics: The Conservation Laws

Mass conservation:

divergence theorem

$$\frac{d}{dt}\int_V \rho \; dV = -\oint_A \rho \boldsymbol{u}\cdot\hat{\boldsymbol{n}} \; dA = -\int_V \nabla\cdot(\rho\boldsymbol{u}) \; dV$$

rate of change of mass in V = mass flux across the surface

rewrite as: $$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u})\right] dV = 0$$

... but the volume V is completely arbitrary

$$\Rightarrow \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = 0$$

equation of continuity

Fluid Dynamics: The Conservation Laws

Momentum conservation:

$$\frac{d}{dt}\int_V \rho u_i \; dV = -\oint_A \rho u_i u_j \hat{n}_j \; dA - \oint_A P \delta_{ij} \hat{n}_j \; dA + \int_V \rho a_i \; dV$$

rate of change of momentum in V =

momentum flux across the surface (fluid flow)

- effect of pressure on surface

+ effect of forces

applying divergence theorem (surface to volume integrals):

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\rho u_i\right) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = -\frac{\partial P}{\partial x_i} + \rho a_i$$

equation of motion

Fluid Dynamics: The Conservation Laws

Energy conservation:

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\rho u^2 + \rho e\right) dV = -\oint_A \left(\frac{1}{2}\rho u^2 + \rho e\right) \boldsymbol{u}\cdot\hat{\boldsymbol{n}}\ \ dA - \oint_A u_i P \delta_{ij} \hat{n}_j dA + \int_V \rho\, \boldsymbol{u}\cdot\boldsymbol{a}\ \ dV$$

rate of change of energy in V =

energy flux across the surface (due to fluid flow)

- work done by pressure

+ work done by forces

applying divergence theorem (surface to volume integrals):

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \nabla\cdot\left(\left(\frac{1}{2}\rho u^2 + \rho e\right)\boldsymbol{u}\right) = -\nabla\cdot\left(P\boldsymbol{u}\right) + \rho\,\boldsymbol{u}\cdot\boldsymbol{a}$$

energy equation

Fluid Dynamics: The Conservation Laws

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho \boldsymbol{u}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho u_i\right) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = -\frac{\partial P}{\partial x_i} + \rho a_i$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \nabla\cdot\left(\left(\frac{1}{2}\rho u^2 + \rho e\right)\boldsymbol{u}\right) = -\nabla\cdot\left(P\boldsymbol{u}\right) + \rho\,\boldsymbol{u}\cdot\boldsymbol{a}$$

general form of conservation laws:

$$\frac{\partial}{\partial t}\left(\textit{density of quantity}\right) + \nabla\cdot\left(\textbf{\textit{flux}}\ \textit{of quantity}\right) = \textit{sources} - \textit{sinks}$$

Fluid Dynamics: The Conservation Laws

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho \boldsymbol{u}\right) = 0$$

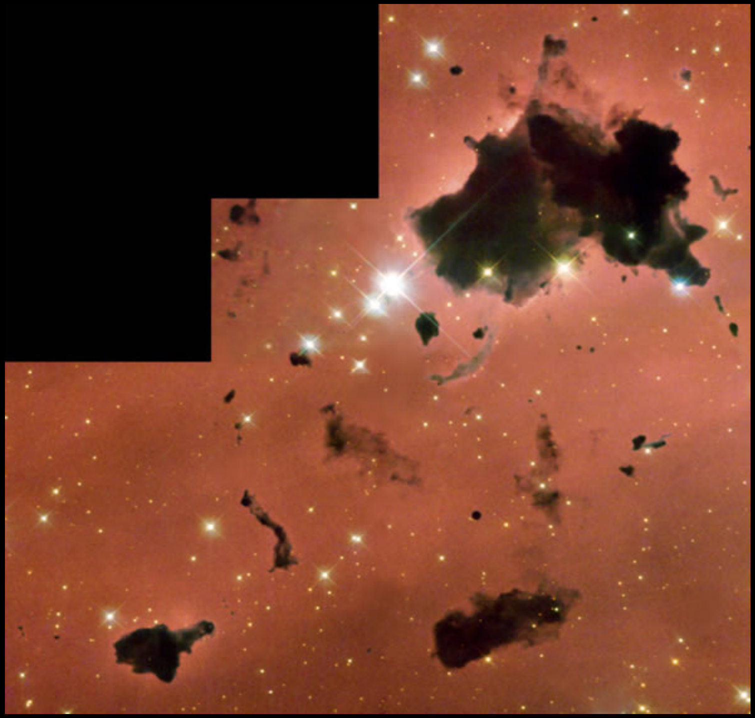
$$\frac{\partial}{\partial t}\left(\rho u_i\right) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = -\frac{\partial P}{\partial x_i} + \rho a_i$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \nabla\cdot\left(\left(\frac{1}{2}\rho u^2 + \rho e\right)\boldsymbol{u}\right) = -\nabla\cdot\left(P\boldsymbol{u}\right) + \rho\,\boldsymbol{u}\cdot\boldsymbol{a}$$

... and what about external forces ?

$$\boldsymbol{a} = \frac{\boldsymbol{F}}{m}$$

Fluid Dynamics: External Forces



Gravitation:

$\boldsymbol{a} = \boldsymbol{g}$ ... acceleration

where $\boldsymbol{g}$ is given by Poisson's equation

$$\nabla \cdot \boldsymbol{g} = -4 \pi G \rho$$

Fluid Dynamics: External Forces

Radiation pressure:

$$f_{rad} = \rho / c \int \kappa_\nu \mathbf{F}_\nu \, d\nu$$

f_{rad} ... force per volume, added to equation of motion

$u \cdot f_{rad}$ ... work done by radiation pressure, added to energy equation

Fluid Dynamics: The Conservation Laws

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho \boldsymbol{u}\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho u_i\right) + \frac{\partial}{\partial x_j}\left(\rho u_i u_j\right) = -\frac{\partial P}{\partial x_i} + \rho g_i + f^i_{rad}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \nabla\cdot\left(\left(\frac{1}{2}\rho u^2 + \rho e\right)\boldsymbol{u}\right) = -\nabla\cdot\left(P\boldsymbol{u}\right) + \rho\,\boldsymbol{u}\cdot\boldsymbol{g}$$

$$+\ \boldsymbol{u}\cdot\boldsymbol{f}_{rad} + \Gamma - \Lambda$$

... including: gravity

radiation pressure

heating & cooling by radiation

Fluid Dynamics: Limit Cases of Energy Transport

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = - \frac{\partial P}{\partial x_i} + \rho a_i$$

Barotropic equation of state: $P = P(\rho)$

Examples: - ideal gas: $P \propto \rho T$

* isothermal case $P \propto \rho$
* adiabatic case $P = K \rho^{\gamma}$

Efficient radiative heating and cooling: $\Gamma - \Lambda = 0 \;\rightarrow\; T \;\rightarrow\; P(\rho, T)$

($\rightarrow$ radiative equilibrium)

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Gas Dynamics: Basic Equations, Waves and Shocks

3. Acoustic Waves and Shock Waves

Fluid Dynamics: The Conservation Laws

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = -\frac{\partial P}{\partial x_i} + \rho a_i$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \nabla \cdot \left(\left(\frac{1}{2}\rho u^2 + \rho e\right)\boldsymbol{u}\right) = -\nabla \cdot (P \boldsymbol{u}) + \rho \boldsymbol{u} \cdot \boldsymbol{a}$$

general form of conservation laws:

$$\frac{\partial}{\partial t}\,(\mathit{density\ of\ quantity}) + \nabla \cdot\,(\boldsymbol{flux}\ \mathit{of\ quantity}) = \mathit{sources} - \mathit{sinks}$$

Small-Amplitude Sound Waves

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) = - \frac{\partial P}{\partial x_i} + \rho a_i$$

replacing energy equation: $P = K \rho^{\gamma}$

1D, no external forces:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u) = - \frac{\partial P}{\partial x}$$

Small-Amplitude Sound Waves

small-amplitude disturbances
in gas which initially is at rest
with constant pressure and density

$$P = P_0 + P_1(x,t)$$
$$\rho = \rho_0 + \rho_1(x,t)$$
$$u = \quad u_1(x,t)$$

replacing energy equation:

$$P = K\rho^{\gamma}$$

1D, no external forces:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u) = -\frac{\partial P}{\partial x}$$

Small-Amplitude Sound Waves

small-amplitude disturbances
in gas which initially is at rest
with constant pressure and density

$$P = P_0 + P_1(x,t)$$
$$\rho = \rho_0 + \rho_1(x,t)$$
$$u = \quad u_1(x,t)$$

insert into equations & linearise:

$$P_1 = \gamma K \rho_0^{\gamma-1} \rho_1 = \gamma \frac{P_0}{\rho_0} \rho_1$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} = 0$$

$$\rho_0 \frac{\partial u_1}{\partial t} = - \frac{\partial P_1}{\partial x}$$

Small-Amplitude Sound Waves

small-amplitude disturbances
in gas which initially is at rest
with constant pressure and density

$$P = P_0 + P_1(x,t)$$
$$\rho = \rho_0 + \rho_1(x,t)$$
$$u = \quad u_1(x,t)$$

$$P_1 = \gamma K \rho_0^{\gamma-1} \rho_1 = \gamma \frac{P_0}{\rho_0} \rho_1$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} = 0$$

use sound speed:

$$\rho_0 \frac{\partial u_1}{\partial t} + a_0^2 \frac{\partial \rho_1}{\partial x} = 0 \qquad \gamma \frac{P_0}{\rho_0} = a_0^2$$

Small-Amplitude Sound Waves

$\Rightarrow$ homogeneous wave equation: $$\frac{\partial^2 \rho_1}{\partial t^2} - a_0^2 \frac{\partial^2 \rho_1}{\partial x^2} = 0$$

general solution: $$\rho_1 = f(x - a_0 t) + g(x + a_0 t)$$

waves propagating with sound speed a_0

$\Longrightarrow$

In the absence of dissipation and spatial inhomogeneities (or dispersion), the waveform of a disturbance governed by a linear wave equation maintains its size and shape forever, apart from propagation at a constant wave speed.

Small-Amplitude Sound Waves

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$\Longrightarrow$ 

In the absence of dissipation and spatial inhomogeneities (or dispersion), the waveform of a disturbance governed by a linear wave equation maintains its size and shape forever, apart from propagation at a constant wave speed.

The Conservation Laws and Burgers' Equation

Fluid equations, 1D:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u + P) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \frac{\partial}{\partial x}\left((\frac{1}{2}\rho u^2 + \rho e + P)u\right) = 0$$

The Conservation Laws and Burgers' Equation

Fluid equations, 1D:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u\right) = 0$$

reformulate the equation of motion:

$$\boxed{\frac{\partial}{\partial t}\left(\rho u\right) + \frac{\partial}{\partial x}\left(\rho u u + P\right) = 0}$$

The Conservation Laws and Burgers' Equation

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reformulate the equation of motion:

$$\boxed{\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u + P) = 0}$$

$$u\frac{\partial \rho}{\partial t} + u\frac{\partial}{\partial x}(\rho u) + \rho\frac{\partial u}{\partial t} + (\rho u)\frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0$$

The Conservation Laws and Burgers' Equation

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$$\rho \frac{\partial u}{\partial t} + (\rho u)\frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0$$

The Conservation Laws and Burgers' Equation

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reformulate the equation of motion:

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$$\rho \frac{\partial u}{\partial t} + (\rho u)\frac{\partial u}{\partial x} + \cancel{\frac{\partial P}{\partial x}} = 0$$

... and neglect pressure gradients:

The Conservation Laws and Burgers' Equation

Fluid equations, 1D:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

reformulate the equation of motion:

$$\boxed{\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u + P) = 0}$$

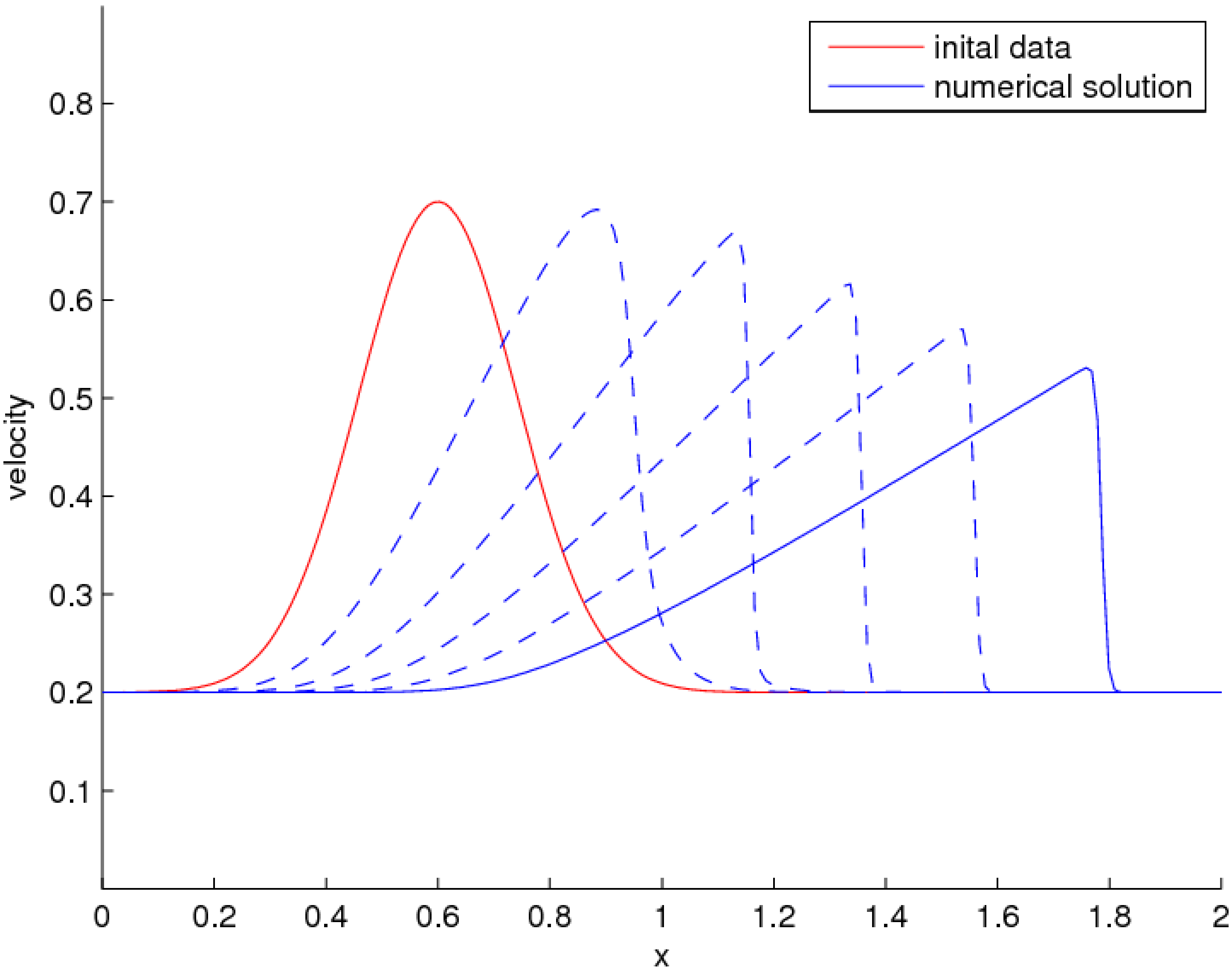
$$\rho\frac{\partial u}{\partial t} + (\rho u)\frac{\partial u}{\partial x} + \cancel{\frac{\partial P}{\partial x}} = 0$$

... and neglect pressure gradients:

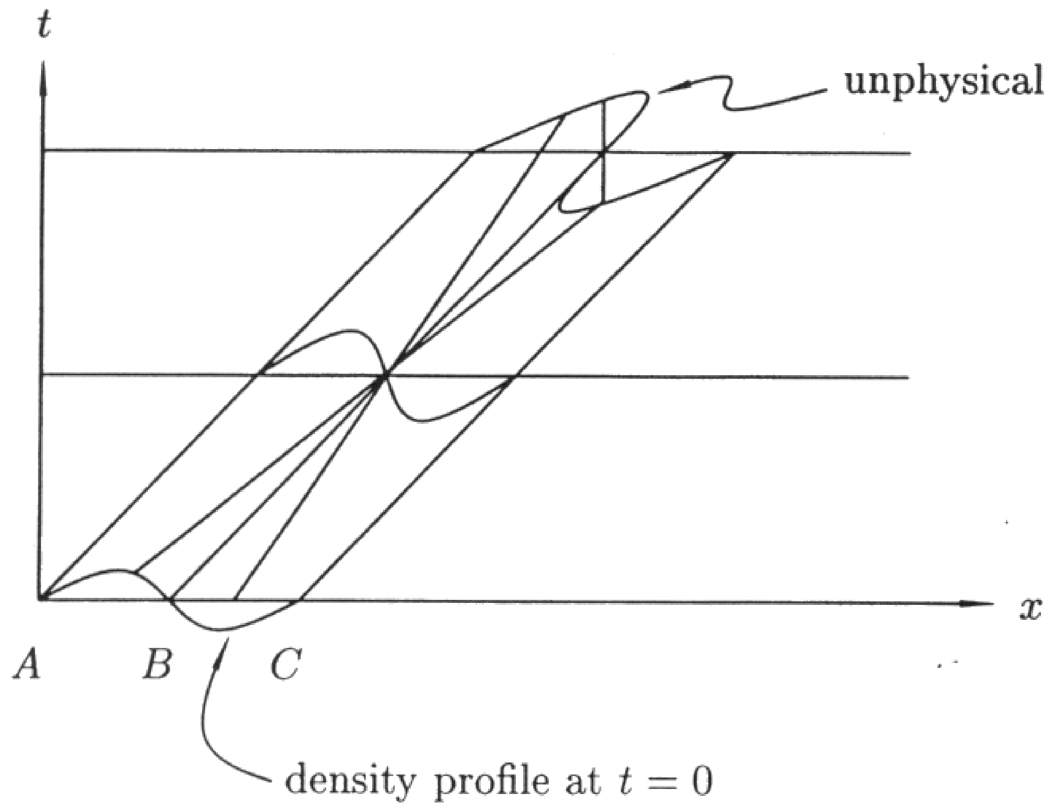
Burgers' equation captures the essential non-linearity of the 1D equation of motion.

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$

Burgers' Equation: Steepening of Waves

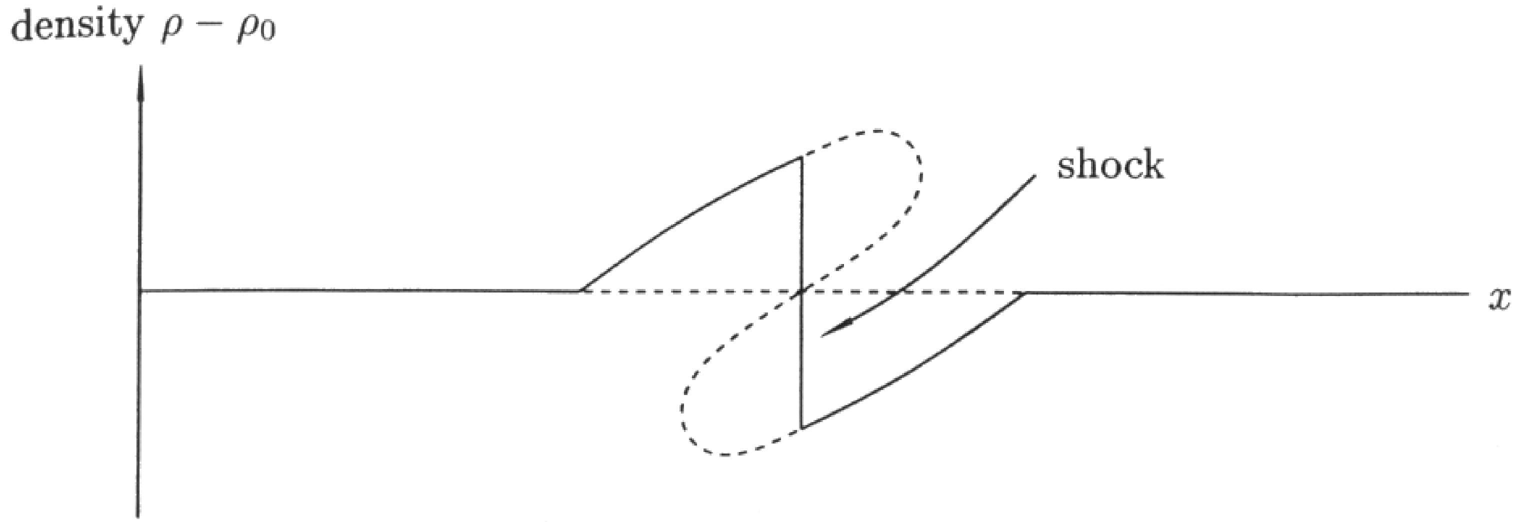


Fluid Dynamics: Steepening of Sound Waves



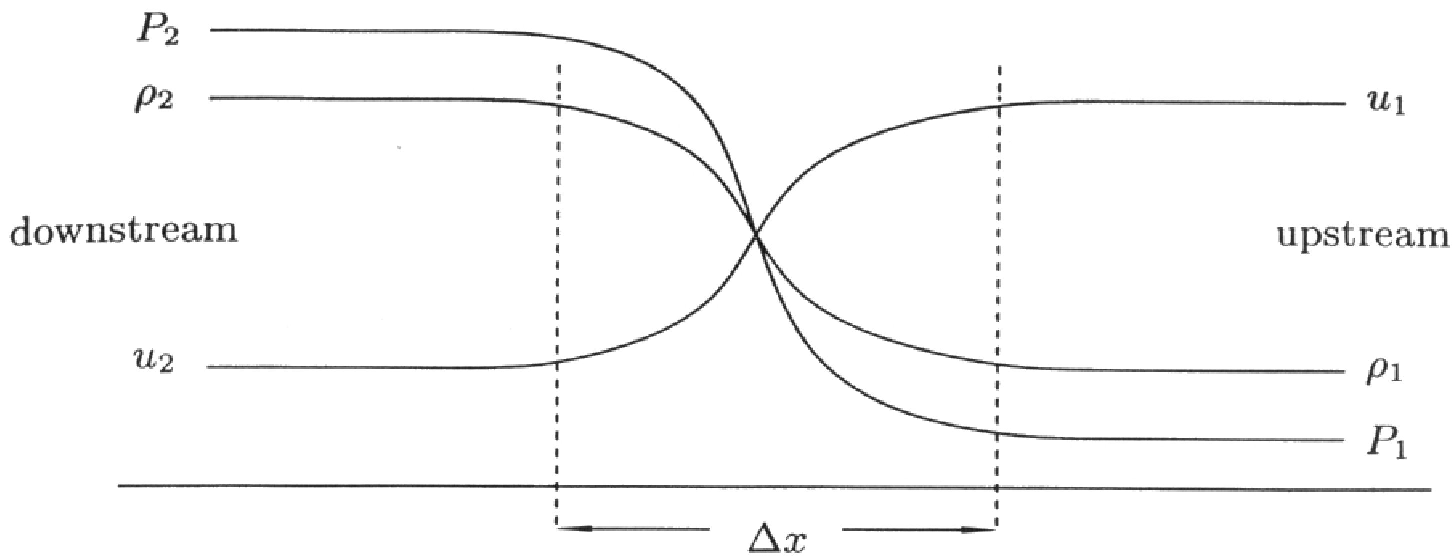
An acoustic wave of finite amplitude, even if it starts with a perfect sinusoidal shape and propagates in an undisturbed medium of exactly uniform properties, would inevitably steepen in its waveform.

Fluid Dynamics: Steepening of Sound Waves



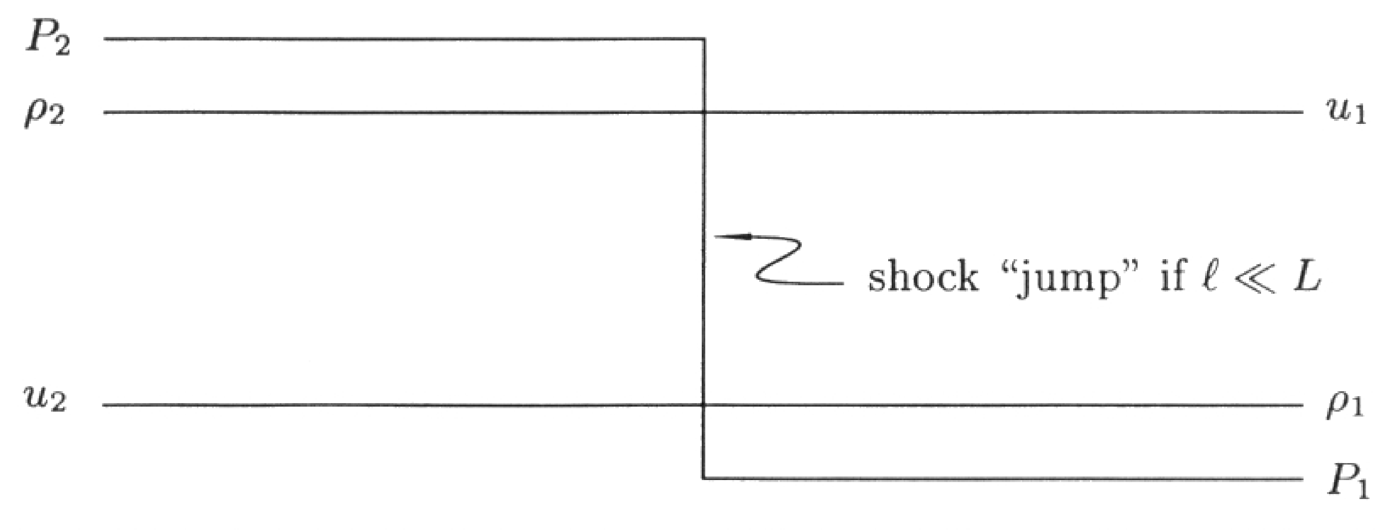
The tendency for nonlinearities to steepen the wave profile, which would produce multiple values for fluid properties such as gas density and velocity, must be eventually offset by the onset of strong viscous forces. The balance of the viscous forces and the steepening tendency mediates a shock, which is approximated in ideal fluid flow as a discontinuous jump of gas properties across the front.

Fluid Dynamics: Structure of Shock Waves



Across a viscous shock, the pressure and density increase and the velocity decreases as the gas flows from the upstream state to the downstream state. The transition is made in a characteristic distance Δx that equals a few mean free paths ℓ for the elastic scattering of the gas particles.

Fluid Dynamics: Structure of Shock Waves



On macroscopic scales, shock transitions may be approximated as single discontinuous jumps.

Shocks and Conservation Laws

1D, no external forces

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho u\right) + \frac{\partial}{\partial x}\left(\rho u u\right) = - \frac{\partial P}{\partial x}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \frac{\partial}{\partial x}\left((\frac{1}{2}\rho u^2 + \rho e)u\right) = - \frac{\partial}{\partial x}\left(P u\right)$$

Shocks and Conservation Laws

1D, no external forces,
stationary flow:

$$\frac{\partial}{\partial x}\left(\rho u\right) = 0$$

$$\frac{\partial}{\partial x}\left(\rho u u\right) = -\frac{\partial P}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\left(\frac{1}{2}\rho u^2 + \rho e\right)u\right) = -\frac{\partial}{\partial x}\left(P u\right)$$

Shocks and Conservation Laws

1D, no external forces,
stationary flow:

$$\frac{\partial}{\partial x}\left(\rho u\right) = 0$$

$$\frac{\partial}{\partial x}\left(\rho u u\right) + \frac{\partial P}{\partial x} = 0$$

$$\frac{\partial}{\partial x}\left(\left(\frac{1}{2}\rho u^2 + \rho e\right)u\right) + \frac{\partial}{\partial x}\left(P u\right) = 0$$

Shocks and Conservation Laws

1D, no external forces, stationary flow:

$$\frac{\partial}{\partial x}\left(\rho u\right) = 0$$

$$\frac{\partial}{\partial x}\left(\rho u u + P\right) = 0$$

$$\frac{\partial}{\partial x}\left(\left(\frac{1}{2}u^2 + e + \frac{P}{\rho}\right) \rho u\right) = 0$$

Shocks and Conservation Laws

1D, no external forces, stationary flow:

$$\frac{\partial}{\partial x}\left(\rho u\right) = 0$$

$$\frac{\partial}{\partial x}\left(\rho u u + P\right) = 0$$

$$\frac{\partial}{\partial x}\left(\left(\frac{1}{2}u^2 + e + \frac{P}{\rho}\right) \rho u\right) = 0$$

jump conditions:

$$\rho_2 u_2 = \rho_1 u_1$$

$$\rho_2 u_2^2 + P_2 = \rho_1 u_1^2 + P_1$$

$$\frac{1}{2}u_2^2 + h_2 = \frac{1}{2}u_1^2 + h_1$$

specific enthalpy

$$h \equiv e + \frac{P}{\rho} = \frac{\gamma}{\gamma - 1}\frac{P}{\rho}$$

Shocks and Conservation Laws

From these relations one can obtain expressions for the ratios of upstream to downstream quantities in terms of the upstream Mach number $M_1 \equiv u_1/a_s$, where $a_s^2 \equiv \gamma P/\rho$. The quantity M_1 is known as the Mach number of the shock. For a perfect gas we find:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_1^2}{(\gamma+1)+(\gamma-1)(M_1^2-1)} = \frac{u_1}{u_2} \tag{75}$$

$$\frac{P_2}{P_1} = \frac{(\gamma+1)+2\gamma(M_1^2-1)}{(\gamma+1)} \tag{76}$$

$$\frac{T_2}{T_1} = \frac{[(\gamma+1)+2\gamma(M_1^2-1)][(\gamma+1)+(\gamma-1)(M_1^2-1)]}{(\gamma+1)^2M_1^2} \tag{77}$$

Notice that $P_2 \geq P_1$, $\rho_2 \geq \rho_1$, and $T_2 \geq T_1$ if $M_1 \geq 1$ (supersonic upstream) with equality if $M_1 = 1$ (no shock at all). In the limit of a very strong shock, $M_1 \to \infty$, the density jump is bounded by a finite value $(\gamma+1)/(\gamma-1)$, which equals 4 if $\gamma = 5/3$. In the same limit, the pressure and temperature jumps have no bound. In any case the deceleration of a gas from supersonic to subsonic speeds in a shock results in compression and heating.