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Gravitational Collapse: Jeans Criterion and Free Fall Time

Lecture Notes

Susanne Höfner

Department of Physics and Astronomy
Uppsala University

Gravitational Collapse and Star Formation

In this course we apply the equations of hydrodynamics to various phases of stellar life. We start with the onset of star formation, asking under which conditions stars can form out of the interstellar medium and which typical time scales govern the gravitational collapse of a cloud.

The Jeans Criterion

We consider a homogeneous gas cloud with given density and temperature, and investigate under which circumstances this configuration is unstable due to self-gravity. For simplicity, we restrict the problem to a one-dimensional analysis. The gas is described by the equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \tag{1}$$

and the equation of motion

$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u) = -\frac{\partial P}{\partial x} - \rho \frac{\partial \Phi}{\partial x} \tag{2}$$

including the gravitational acceleration $\mathbf{g} = -\partial\Phi/\partial x$ The gravitational potential Φ is given by Poisson's equation

$$\frac{\partial^2 \Phi}{\partial x^2} = 4\pi G \rho \tag{3}$$

describing the self-gravity of the gas (G is the constant of gravitation).

We assume that the gas is isothermal and replace the energy equation by a barotropic equation of state

$$P = c_s^2 \rho \tag{4}$$

where c_s is the isothermal sound speed. This assumption is justified by the fact that the energy exchange by radiation is very efficient for typical interstellar matter, i.e. the time scales for thermal adjustment are short compared to the dynamical processes we study here.

We assume that initially the gas has a constant density ρ_0, a constant pressure P_0 and is at rest ($u_0 = 0$). The corresponding gravitational potential Φ_0 follows from Eq. 3. We consider a small perturbation such that

$$\rho = \rho_0 + \rho_1 \,, \qquad P = P_0 + P_1 \,, \qquad u = u_1 \,, \qquad \Phi = \Phi_0 + \Phi_1 \tag{5}$$

and all quantities with index '1' (depending on space and time) are small compared to the corresponding quantities with index '0'. In addition, we assume the perturbation itself to be isothermal so that c_s remains unchanged and

$$P_1 = c_s^2 \, \rho_1 \,. \tag{6}$$

Inserting the relations 5 into Eqs. 1, 2 and 3 and linearizing these equations we obtain

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} = 0 \tag{7}$$

$$\frac{\partial u_1}{\partial t} = -\frac{\partial \Phi_1}{\partial x} - \frac{c_s^2}{\rho_0}\frac{\partial \rho_1}{\partial x} \tag{8}$$

$$\frac{\partial^2 \Phi_1}{\partial x^2} = 4\pi G \rho_1 \tag{9}$$

This is a linear homogeneous system of differential equations for ρ_1, u_1 and Φ_1 with constant coefficients and we can assume that solutions exist with the space and time dependence proportional to $\exp[\mathrm{i}(kx + \omega t)]$. Therefore,

$$\frac{\partial}{\partial x} = \mathrm{i}k\,, \qquad \frac{\partial}{\partial t} = \mathrm{i}\omega \tag{10}$$

and we obtain

$$\omega \rho_1 + k\rho_0 u_1 = 0 \tag{11}$$

$$\frac{k\,c_s^2}{\rho_0}\rho_1 + \omega u_1 + k\Phi_1 = 0 \tag{12}$$

$$4\pi G\rho_1 + k^2\Phi_1 = 0\,. \tag{13}$$

This homogeneous linear system of equations for ρ_1, u_1 and Φ_1 can only have non-trivial solutions if the determinant

$$\begin{vmatrix} \omega & k\rho_0 & 0 \\ \frac{k\,c_s^2}{\rho_0} & \omega & k \\ 4\pi G & 0 & k^2 \end{vmatrix} \tag{14}$$

is zero, i.e. if

$$\omega^2 = k^2 c_s^2 - 4\pi G\rho_0\,. \tag{15}$$

Assuming a non-vanishing wavenumber k we can distinguish two different cases:

- If k is sufficiently large then $k^2c_s^2 - 4\pi G\rho_0 > 0$ and ω is real. The perturbation varies periodically in time and the equilibrium is stable with respect to this perturbation (the amplitude does not increase with time).[1]

- If $k^2c_s^2 - 4\pi G\rho_0 < 0$ then ω is of the form $\mathrm{i}\zeta$ where ζ is real. Therefore there exist perturbations which grow exponentially with time, i.e. the equilibrium is unstable.

The border between this two regimes corresponds to a critical wavenumber

$$k_{\mathrm{J}} = \left(\frac{4\pi G\rho_0}{c_s^2}\right)^{1/2} \tag{16}$$

[1] In the limit $k \to \infty$ we obtain $\omega^2 = k^2 c_s^2$ which corresponds to isothermal sound waves.

or to a critical wavelength ($\lambda = 2\pi/k$)

$$\lambda_\mathrm{J} = \left(\frac{\pi}{G\rho_0}\right)^{1/2} c_s\,. \tag{17}$$

Therefore, a perturbation with a wavelength $\lambda > \lambda_\mathrm{J}$ (i.e. $k < k_\mathrm{J}$) is unstable. This condition is called the *Jeans criterion* after James Jeans who derived it in 1902.

Jeans Mass

Using the density ρ_0 and (half of) the wavelength λ_J as a length scale we can define a critical mass

$$M_\mathrm{J} = \rho_0 \left(\frac{\lambda_\mathrm{J}}{2}\right)^3 \tag{18}$$

which is called Jeans mass. With the definition of the isothermal sound speed and the equation of state of an ideal gas

$$c_s = \left(\frac{P_0}{\rho_0}\right)^{1/2} = \left(\frac{k}{\mu\, m_u}\, T_0\right)^{1/2} \tag{19}$$

where k is the Boltzmann constant, μ the mean molecular weight and $m_u = 1.66 \cdot 10^{-27}\,\mathrm{kg}$ the atomic mass unit, we obtain

$$M_\mathrm{J} = \rho_0 \left(\frac{\pi k\, T_0}{4\, \mu\, m_u G\, \rho_0}\right)^{3/2} \propto T_0^{3/2} \rho_0^{-1/2} \tag{20}$$

A gas cloud with given temperature T_0 and density ρ_0 and a mass larger than the Jeans mass is unstable and will eventually collapse due to its own gravitation.

During the initial phases of the collapse the gas will stay isothermal as long as the matter is more or less optically thin and the energy exchange by radiation is efficient. In these phases the Jeans mass decreases with the increasing density, i.e. smaller and smaller parts of the original cloud may become instable, leading to fragmentation and the formation of a group of less massive stars. As the density increases, the collapsing clouds finally become optically thick and the heat resulting from the compression can not be radiated away any more. This leads to an increase of the temperature which finally causes the Jeans mass to increase again, stopping further fragmentation.

Free-fall Collapse of a Homogeneous Cloud

In the previous section we have investigated under which conditions an interstellar gas cloud may become unstable and collapse under the influence of its own gravitation. Now we want to know what the typical timescales for the collapse are.

We consider a spherically symmetric, homogeneous collapsing cloud with mass M and radius R, assuming free fall, i.e. neglecting the forces due to pressure gradients. Since the gravitational force is $\approx (GM)/R^2$ and the pressure term in the equation of motion can be approximated by

$$\left|\frac{1}{\rho}\frac{\partial P}{\partial R}\right| \approx \frac{P}{\rho R} \approx \frac{kT}{\mu\, m_u\, R} \tag{21}$$

the ratio of the gravitation term to pressure term is $\propto M/(RT)$, and increases during the isothermal phases of the collapse with the decreasing radius R of the cloud (M is constant).

We follow the evolution of a free-falling spherical mass shell with radius r in a co-moving (Lagrangian) frame of reference. The equation of motion (free fall) is

$$\frac{d^2 r}{dt^2} = -\frac{Gm}{r^2} \tag{22}$$

where m denotes the total mass contained within radius r. Introducing the velocity of the shell

$$u(r(t)) = \frac{dr}{dt} \tag{23}$$

we can re-write the left-hand side of the equation of motion as

$$\frac{d^2 r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(u(r)\right) = \frac{dr}{dt}\frac{d}{dr}u = u\frac{d}{dr}u = \frac{1}{2}\frac{d}{dr}u^2 \tag{24}$$

and the equation of motion as

$$\frac{1}{2}d(u^2) = -\frac{Gm}{r^2}dr\,. \tag{25}$$

By integration we obtain

$$u^2 = 2GM\left(\frac{1}{r} - \frac{1}{R}\right) \tag{26}$$

where the constant of integration (second term on the right-hand side) has been chosen such that the initial conditions $u(t=0) = 0$, $r(t=0) = R$ and $m = M$ are fulfilled. Using $u = dr/dt$ we find

$$\frac{dr}{dt} = \pm\left[2GM\left(\frac{1}{r} - \frac{1}{R}\right)\right]^{1/2} \tag{27}$$

or (chosing the solution with $u < 0$)

$$dt = \frac{-dr}{\left[2GM\left(\frac{1}{r} - \frac{1}{R}\right)\right]^{1/2}} = \frac{-dr}{\left(\frac{2GM}{R}\right)^{1/2}\left(\frac{R}{r} - 1\right)^{1/2}}\,. \tag{28}$$

With the substitution

$$\xi = \frac{r}{R} \qquad d\xi = \frac{dr}{R} \tag{29}$$

we obtain

$$dt = -\left(\frac{2GM}{R^3}\right)^{-1/2}\frac{d\xi}{\left(\frac{1}{\xi} - 1\right)^{1/2}} = -\left(\frac{8\pi G\rho_0}{3}\right)^{-1/2}\left(\frac{\xi}{1-\xi}\right)^{1/2} d\xi \tag{30}$$

where we have used $M/R^3 = 4\pi\rho_0/3$ and ρ_0 denotes the initial density of the cloud. Integrating the right-hand side from $\xi = 1$ to $\xi = 0$ (i.e. $r = R$ to $r = 0$) we obtain the so-called free-fall time $t_{\rm ff}$ of the cloud,

$$t_{\rm ff} = -\left(\frac{8\pi G\rho_0}{3}\right)^{-1/2} \int_{\xi=1}^{0} \left(\frac{\xi}{1-\xi}\right)^{1/2} d\xi \tag{31}$$

i.e. the time it takes until the (pressureless) cloud has contracted to a point (or a radius which is much smaller than its original radius R). The integral can be calculated analytically (e.g. by using the substitution $\xi = \sin^2\phi$) , and we find

$$t_{\rm ff} = \left(\frac{3\pi}{32G\rho_0}\right)^{1/2} . \tag{32}$$

References

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