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Gas Dynamics: Basic Equations, Waves and Shocks

Lecture Notes

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1 The Equations of Fluid Dynamics

In the following, we discuss the basic equations which describe the flow of a gas from a macroscopic point of view. We will derive and interpret the equations by looking at the changes of macroscopic quantities within a fixed volume. Considering ideal fluids only, we neglect the effects of viscosity and heat conduction which are of little importance in many astrophysical flows. Furthermore, we will discuss relevant external forces and energy exchange processes.

1.1 The Conservation Laws

Consider an arbitrary but fixed volume V with a surface A (with local unit normal $\hat{\mathbf{n}}$): the time rate of change of a quantity inside this volume will be given by the sum of explicit changes of this quantity (volumetric contributions) and surface effects (net transport across the surface).

Mass Conservation

The time rate of change of the mass contained in the volume V is equal to the (negative) value of the mass flux $\rho\,\mathbf{u}$ across the surface (with ρ and $\mathbf{u}$ denoting the mass density and flow velocity, respectively):

$$\frac{d}{dt}\int_V \rho \; dV = -\oint_A \rho\,\mathbf{u}\cdot\hat{\mathbf{n}}\,dA = -\int_V \nabla\cdot(\rho\,\mathbf{u}) \; dV \tag{1}$$

where the last expression is obtained by using the divergence theorem. V is fixed, so we can write

$$\int_V \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{u})\right] \; dV = 0 \tag{2}$$

and, since V is completely arbitrary, the integrand must vanish. Thus we obtain the equation of continuity:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\,\mathbf{u}) = 0. \tag{3}$$

Momentum Conservation

The time rate of change of the fluid momentum in volume V equals the surface integral of the momentum flux due to fluid flow across A, plus the effect of the pressure P acting on the fluid across the surface A, plus the contribution of external forces (e.g. gravity) acting on every point of V (causing an acceleration $\mathbf{a}$):

$$\frac{d}{dt}\int_V \rho\,u_i \; dV = -\oint_A (\rho\,u_i)\,u_j\hat{n}_j\,dA - \oint_A P\,\delta_{ij}\hat{n}_j\,dA + \int_V \rho\,a_i \; dV \tag{4}$$

Applying the divergence theorem to transform the surface integrals into volume integrals we obtain the momentum equation

$$\frac{\partial}{\partial t}(\rho\,u_i) + \frac{\partial}{\partial x_j}(\rho\,u_i u_j) = -\frac{\partial P}{\partial x_i} + \rho\,a_i \tag{5}$$

Energy Conservation

The time rate of change of the total fluid energy (kinetic energy of fluid motion plus internal energy, where e denotes the specific internal energy) equals the surface integral of the energy flux (kinetic + internal), plus the surface integral of the work done by the pressure plus the volume integral of the work done by external forces (e.g. gravitation):

$$\frac{d}{dt}\int_V \left[\frac{1}{2}\rho u^2 + \rho e\right] \, dV = -\oint_A \left[\frac{1}{2}\rho u^2 + \rho e\right] \mathbf{u}\cdot\hat{\mathbf{n}}\, dA - \oint_A u_i P\, \delta_{ij}\hat{n}_j \, dA + \int_V \rho\, \mathbf{u}\cdot\mathbf{a}\, dV \quad (6)$$

Using the divergence theorem we obtain the total energy equation:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho u^2 + \rho e\right] + \nabla\cdot\left(\left[\frac{1}{2}\rho u^2 + \rho e\right] \mathbf{u}\right) = -\nabla\cdot(P\,\mathbf{u}) + \rho\,\mathbf{u}\cdot\mathbf{a} \quad (7)$$

We see that the fluid equations have the general form

$$\frac{\partial}{\partial t}\,(\text{density of quantity}) + \nabla\cdot(\textbf{flux} \text{ of quantity}) = \text{sources} - \text{sinks}.$$

1.2 External Forces and Radiation

So far, we have taken external forces formally into account by considering the acceleration $\mathbf{a}$ acting on the gas. In most astrophysical situations gravity is the force which determines the large-scale structure of an object. In this case we have $\mathbf{a} = \mathbf{g}$ where $\mathbf{g}$ is the the gravitational acceleration, given by Poisson's equation (G = constant of gravitation)

$$\nabla\cdot\mathbf{g} = -4\pi G\rho. \quad (8)$$

In some circumstances we can not ignore the interaction of the gas with a radiation field. The force per unit volume exerted by the radiation on the gas is

$$\mathbf{f}_{\text{rad}} = \frac{\rho}{c}\int_0^\infty \kappa_\nu \mathbf{F}_\nu \, d\nu, \quad (9)$$

were ν denotes the frequency, κ_ν is the opacity of the gas (absorption and scattering) and $\mathbf{F}_\nu$ is the monochromatic radiative energy flux. In this case we have to add

$$\mathbf{f}_{\text{rad}}$$

to the right-hand side of Eq. 5, and account for the work done by this force by adding

$$\mathbf{u}\cdot\mathbf{f}_{\text{rad}}$$

to the right-hand side of Eq. 7. In addition, we also have to take radiative sources and sinks of heat (internal energy) into account by adding

$$\Gamma - \Lambda$$

to the right-hand side of Eq. 7. Here Γ and Λ denote the volumetric gains and losses of energy due to local sources and sinks (absorption and emission of radiation), respectively.

Note in this context that this simple treatment of the interaction of radiation and matter can only be used if the flow velocity is much smaller than the speed of light.

1.3 Limit Cases of the Energy Transport

Depending on which processes are relevant for the transport of energy in the system under consideration, the energy equation can be rather complicated. However, in many situations this equation can be replaced by a barotropic equation of state, i.e. a relation

$$P = P(\rho). \tag{10}$$

One example for such a case is a fluid consisting of degenerate matter. Two other examples are the limit cases of negligible and extremely efficient energy transport.

Adiabatic Flow

Consider an element of gas which neither gains nor loses heat by contact with its surroundings (no thermal conduction, radiative energy exchange, etc.) and the internal energy only changes due to work performed by the element on its surroundings, and vice versa. Then the entropy of this element remains constant and from thermodynamics we know that the pressure and temperature of the gas are related by the adiabatic relationship

$$P = K\,\rho^{\gamma}.$$

Here $\gamma = C_P/C_V$ is the ratio of the heat capacities (e.g. $\gamma = 5/3$ for a monatomic gas) and K is a constant related to the entropy of the element.

Radiative Equilibrium

On the other hand, in the limit of extremely efficient radiative heating and cooling the energy equation is dominated by the radiative source and sink terms (see preceding section) and reduces to

$$\Gamma - \Lambda \equiv 0.$$

For a gas of given composition under optically thin conditions $\Gamma - \Lambda$ can often be expressed as a function of density and temperature, only. Then $\Gamma - \Lambda \equiv 0$ associates a unique temperature with a given density, which implies, together with the equation of state for an ideal gas, that $P = P(\rho)$.

References

A.R. Choudhuri, *The Physics of Fluids and Plasmas, An Introduction for Astrophysicists,* Cambridge University Press, 1998, p. 53-58, 61-62

F.H. Shu, *The Physics of Astrophysics, Vol.II: Gas Dynamics,* University Science Books, 1992, p. 44-53, 64-65

2 Acoustic Waves and Shock Waves

2.1 Small-amplitude Acoustic Waves

We consider the propagation of acoustic disturbances (sound waves) in an ideal gas where the pressure and density are related by

$$P = K\,\rho^{\gamma} \tag{11}$$

(see preceding section for definitions). In the following, Eq. 11 replaces the energy conservation equation.

We assume that initially the gas is at rest, has a constant pressure P_0 and a constant density ρ_0, and no external forces are acting on the gas. Now we consider a small disturbance so that the pressure, density and velocity are:

$$P = P_0 + P_1(x,t) \tag{12}$$

$$\rho = \rho_0 + \rho_1(x,t) \tag{13}$$

$$u = u_1(x,t). \tag{14}$$

The perturbations P_1, ρ_1 and u_1 are assumed to be of small amplitude so that we can neglect terms of higher order than linear in these quantities. Linearizing the (one-dimensional) equations of mass and momentum conservation (acceleration due to external forces $\mathbf{a} = 0$), as well as Eq. 11 we obtain

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} = 0 \tag{15}$$

$$\rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial P_1}{\partial x} \tag{16}$$

and

$$P_1 = \gamma\, K\, \rho_0^{\gamma-1} \rho_1 = \gamma\, \frac{P_0}{\rho_0}\, \rho_1. \tag{17}$$

We define the quantity a_0^2 by

$$a_0^2 = \gamma\, \frac{P_0}{\rho_0}. \tag{18}$$

where a_0 has the dimension of velocity and is constant. Using this definition, the linearized momentum equation can be written as

$$\rho_0 \frac{\partial u_1}{\partial t} + a_0^2 \frac{\partial \rho_1}{\partial x} = 0. \tag{19}$$

Differentiation of Eq. 15 with respect to t, of Eq. 19 with respect to x and subtraction of the resulting equations gives

$$\frac{\partial^2 \rho_1}{\partial t^2} - a_0^2 \frac{\partial^2 \rho_1}{\partial x^2} = 0. \tag{20}$$

This is the homogeneous wave equation which has as its most general solution

$$\rho_1 = f(x - a_0 t) + g(x + a_0 t), \tag{21}$$

where f and g are waves propagating in opposite directions with speed a_0 (adiabatic sound speed), maintaining their original shape.

2.2 Shock Waves

In the previous section we have considered small disturbances which are described by a linear wave equation. In general, due to the non-linear nature of the hydrodynamic equations, waves with a finite amplitude will not maintain their shape as they propagate. On the contrary, such waves have a tendency to steepen, because different parts of the wave move with different speeds. The limit of this steepening process is a shock wave, i.e. a 'discontinuous' change of the velocity, density and pressure at the location of the shock.

We consider a stationary shock, i.e. the velocity, density and pressure at either side of the shock do not explicitly depend on time, and we choose the frame of reference in which the shock is at rest. According to the conservation equations for mass, momentum and energy the flow across the shock front has to satisfy the following *jump conditions*:

$$\begin{aligned} \rho_2\, u_2 &= \rho_1\, u_1 && (22) \\ \rho_2\, u_2^2 + P_2 &= \rho_1\, u_1^2 + P_1 && (23) \\ \frac{1}{2}\, u_2^2 + h_2 &= \frac{1}{2}\, u_1^2 + h_1 && (24) \end{aligned}$$

These equations give the downstream conditions (index 2) as a function of the upstream conditions (index 1). They are known as the *Rankine-Hugoniot jump conditions*. Here h denotes the specific enthalpy which for a perfect gas satisfies the constitutive relations

$$h = \frac{\gamma}{\gamma - 1}\, \frac{P}{\rho} = \frac{\gamma}{\gamma - 1}\, \frac{kT}{\mu\, m_u} \tag{25}$$

where k, μ and m_u are the Boltzmann constant, the mean molecular weight and the atomic mass unit, respectively. From these relations one can obtain expressions for the ratios of upstream to downstream quantities in terms of the upstream Mach number $M_1 \equiv u_1/a_s$, where $a_s \equiv \sqrt{\gamma P/\rho}$ (adiabatic sound speed). For an ideal gas one finds:

$$\begin{aligned} \frac{\rho_2}{\rho_1} &= \frac{(\gamma+1)M_1^2}{(\gamma+1) + (\gamma-1)(M_1^2 - 1)} = \frac{u_1}{u_2} && (26) \\ \frac{P_2}{P_1} &= \frac{(\gamma+1) + 2\gamma(M_1^2 - 1)}{(\gamma+1)} && (27) \\ \frac{T_2}{T_1} &= \frac{[(\gamma+1) + 2\gamma(M_1^2 - 1)][(\gamma+1) + (\gamma-1)(M_1^2 - 1)]}{(\gamma+1)^2 M_1^2} && (28) \end{aligned}$$

Note that $P_2 \geq P_1$, $\rho_2 \geq \rho_1$, and $T_2 \geq T_1$ if $M_1 \geq 1$ (supersonic upstream) with equality if $M_1 = 1$ (no shock at all). In the limit of a very strong shock, $M_1 \to \infty$, the density jump is bounded by a finite value $(\gamma+1)/(\gamma-1)$, which equals 4 if $\gamma = 5/3$. In the same limit, the pressure and temperature jumps have no bound. In any case the deceleration of a gas from supersonic to subsonic speeds in a shock results in compression and heating.

References

A.R. Choudhuri, *The Physics of Fluids and Plasmas,* p. 106-109, 111-116

F.H. Shu, *The Physics of Astrophysics, Vol.II: Gas Dynamics,* p. 203-216