Planetary System Dynamics

Part I

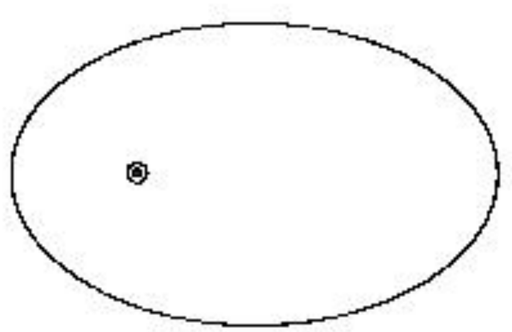
Kepler's Laws (1609-1619)

- Law I. *The orbit of each planet is an ellipse, with the Sun at one of its foci.*
- Law II. *As the planet moves in its orbit, the radius vector sweeps out equal areas in equal intervals of time.*
- Law III. *For any two planets, the ratio of the squares of their periods of revolution about the Sun is the same as the ratio of the cubes of their mean distances from the Sun.*

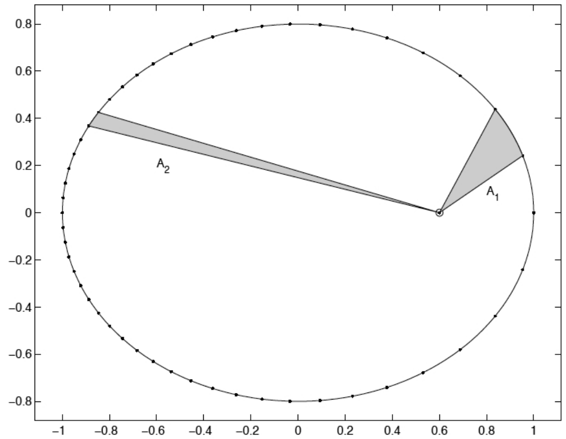
The first nearly correct general description of planetary motions

Empirical fit to the observations, no causal explanation

Kepler I



Kepler II



Kepler III

$$P^2 \propto a^3$$

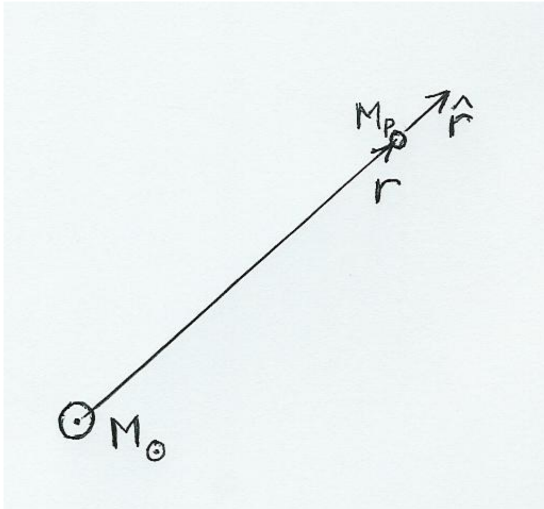
Inaccuracies

- ***Why do Kepler's Laws not give an exact description of the real planetary orbits?***

- The planetary masses enter into the 3rd Law
- The planets perturb each others' orbits
- Newtonian theory is not exact (GR effects)

Newton's law of gravity

$$\ddot{r} = -\frac{G(M_{sun} + M_p)}{|r|^2}\hat{r}$$



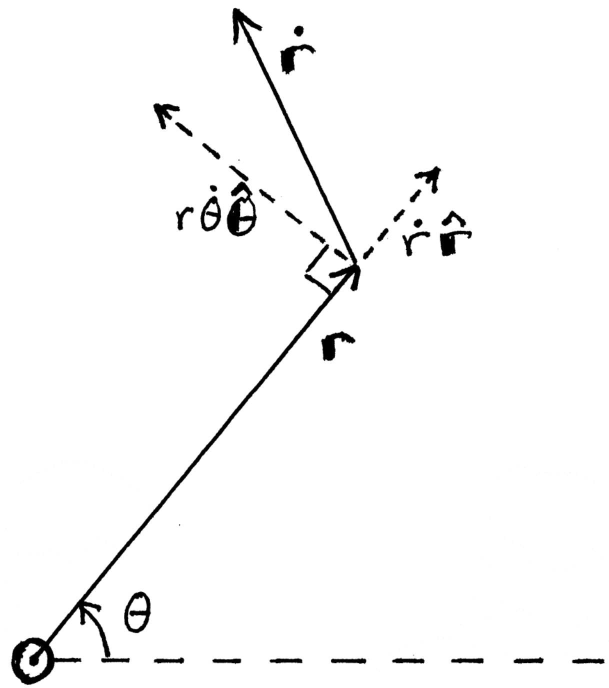
- Conservative force field ⇒ ***Energy integral***
- Central force field ⇒ ***Angular momentum integral***
- Solution: Conic section satisfying Kepler's laws

The velocity vector

- **Radial and transverse components:**

- **Radial** - Magnitude = time derivative of r

- **Transverse** - Magnitude = r times time derivative of θ



Polar coordinates:
r *and* θ

Basic integrals (1)

Point mass potential (energy per unit mass): $U(r) = -\dfrac{GM}{r}$

velocity: $$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta}$$

Energy integral: $$\frac{1}{2}\dot{\mathbf{r}}^2 + U(r) = E$$

$$\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) = E$$

Basic integrals (2)

Angular momentum integral: $\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{h}$ *(constant vector)*

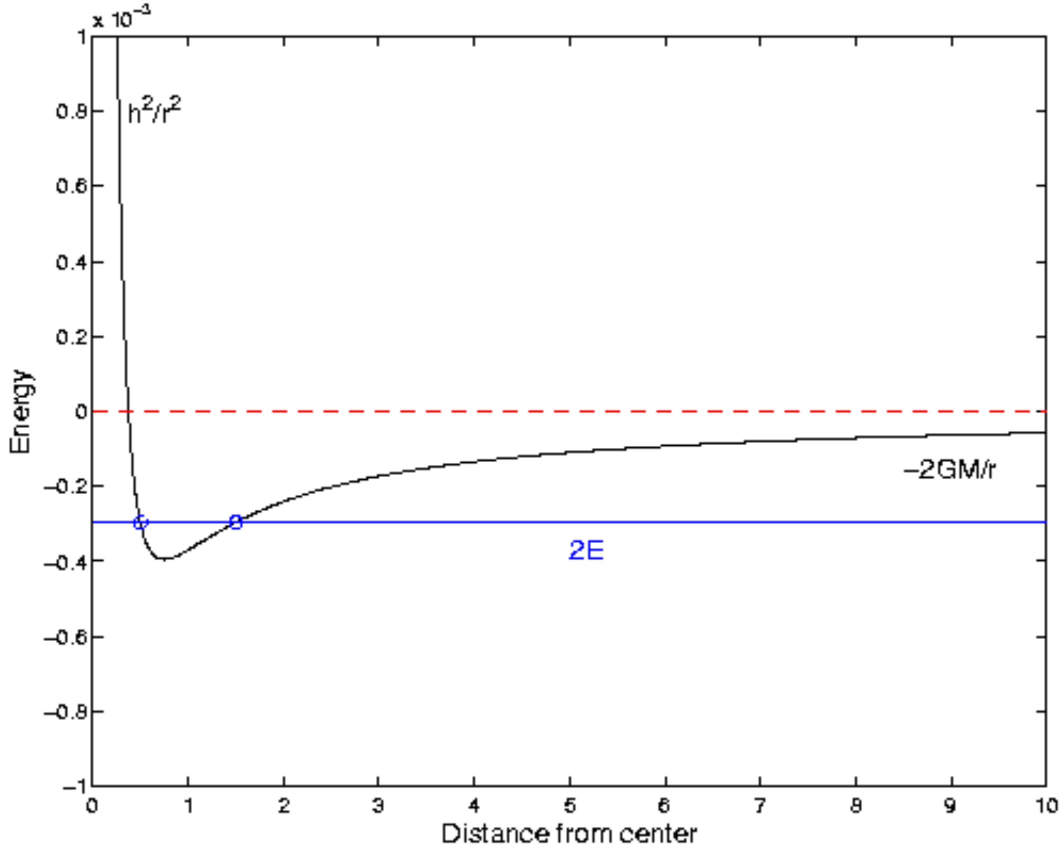
$$\mathbf{h} = r^2 \dot{\theta} \, (\hat{\mathbf{r}} \times \hat{\theta});$$

$$h = r^2 \dot{\theta}$$ *(constant)*

$$\dot{r}^2 + \frac{h^2}{r^2} - \frac{2GM}{r} = 2E$$

Properties of motion

$$f(r) = \frac{h^2}{r^2} - \frac{2GM}{r}$$ must not be larger than 2*E!*



Bound motion in a limited interval of r

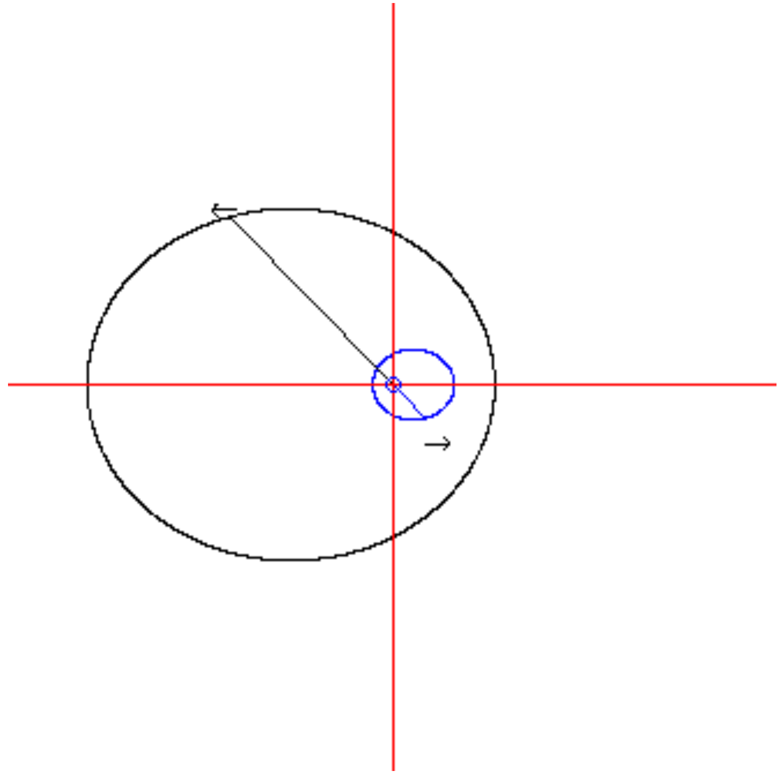
Reflex motion

The hammer and the thrower both rotate around their common center of mass, but the hammer moves much faster at a larger distance

The same happens with a planet and its star

Stellar/Planetary orbits

- The Center of Mass (CM) is fixed in the origin, and the star-planet radius vector turns around the CM
- Equations of motion in the CM frame:

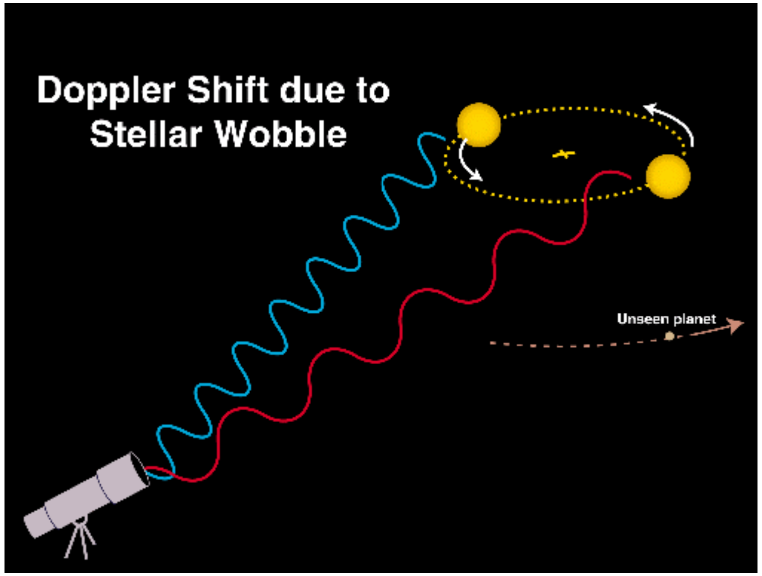


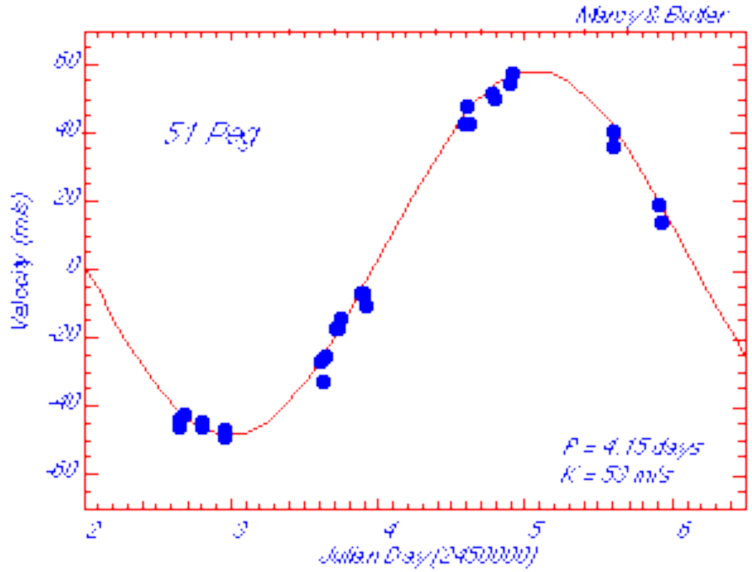
$$\ddot{r}_p = -\frac{GM_*}{|r|^2}\hat{r} \qquad \ddot{r}_* = \frac{GM_p}{|r|^2}\hat{r}$$

Subtract to get the equation of relative motion!

Exoplanets

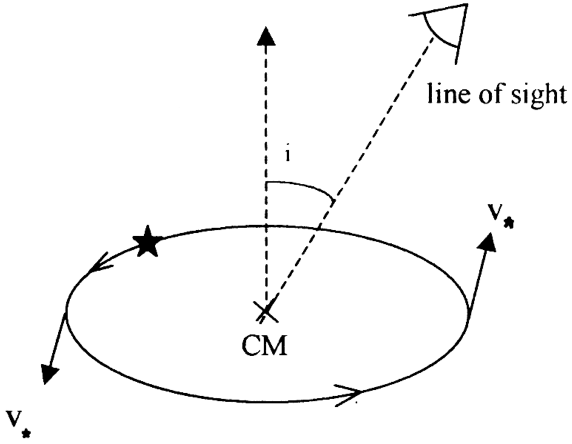
M. Mayor, G. Marcy

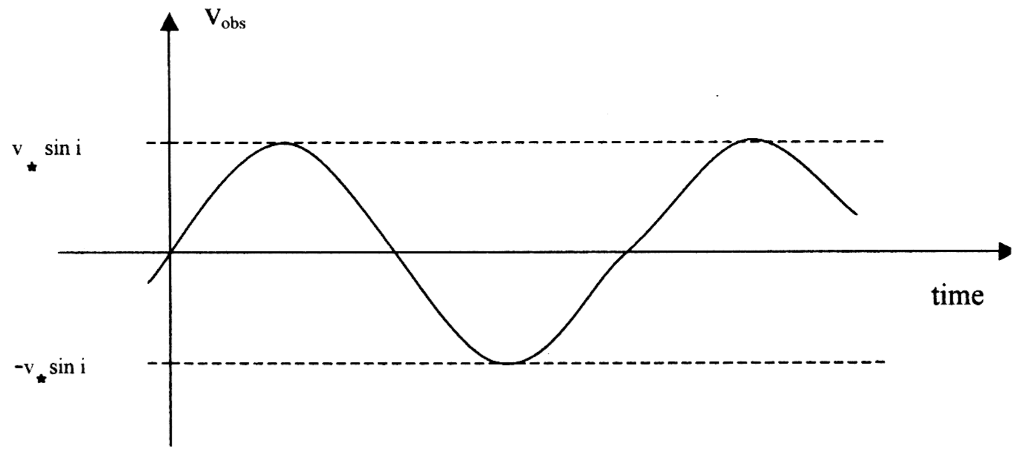




- radial motions of solar type stars are measured from the Doppler shifts of a large number of narrow absorption lines in the optical spectrum
- useful to a distance of about 160 light years
- The large majority of exoplanets were detected by this method

Exoplanet mass determination





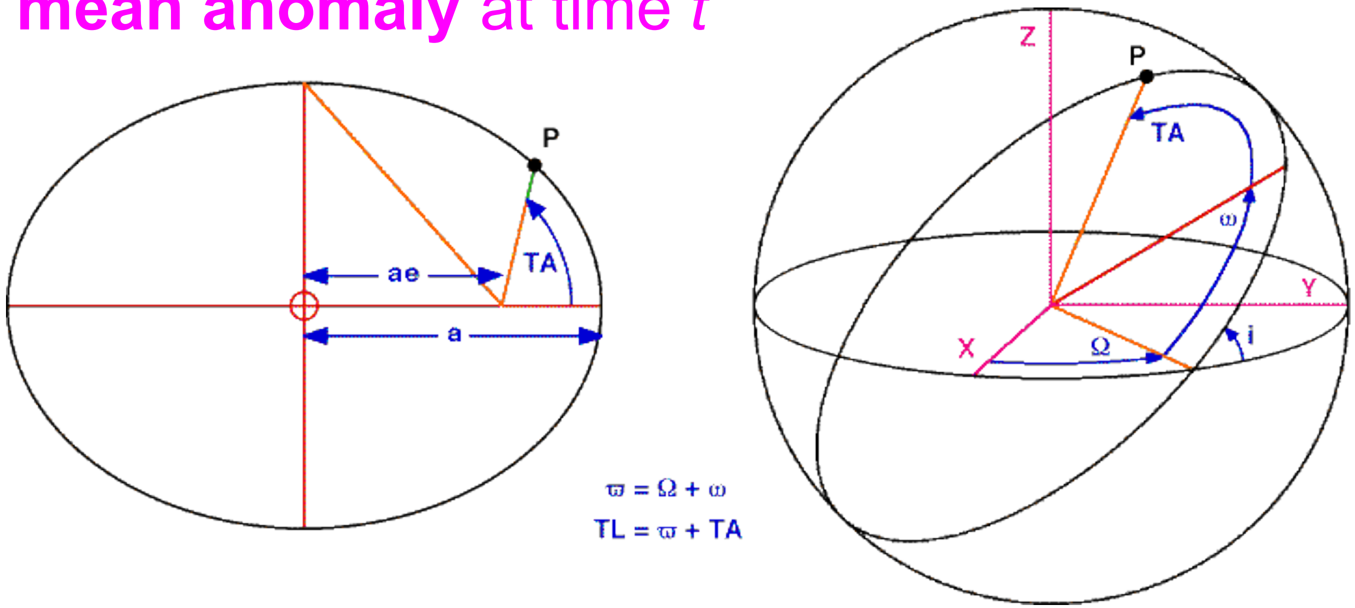
- Measure: period *P* and radial velocity half amplitude $v_* \sin i$
- Identifying v_* with $v_* \sin i$ yields a lower limit for M_p
- Detection is easiest if *P* is short (v_p is large), M_p is large, or M_* is small

$$M_p = M_* \times \frac{v_*}{v_p} \qquad v_* \propto M_p\, P^{-1/3}\, M_*^{-2/3}$$

Orbital Elements

- *a* - semimajor axis
- *e* - eccentricity
- *i* - inclination w.r.t. the ecliptic
- Ω - longitude of the ascending node
- ω - argument of perihelion
- *T* - time of perihelion passage

M=n(t-T) is **mean anomaly** at time *t*



Useful Relations

- Perihelion distance $q = a(1 - e)$
- Aphelion distance $Q = a(1 + e)$

- Binding energy $E = -\dfrac{GM_{sun}}{2a}$
- Speed of motion $v^2 = GM_{sun}\left(\dfrac{2}{r} - \dfrac{1}{a}\right)$
- Angular momentum $H = \sqrt{GM_{sun}a(1 - e^2)}$

Critical velocities

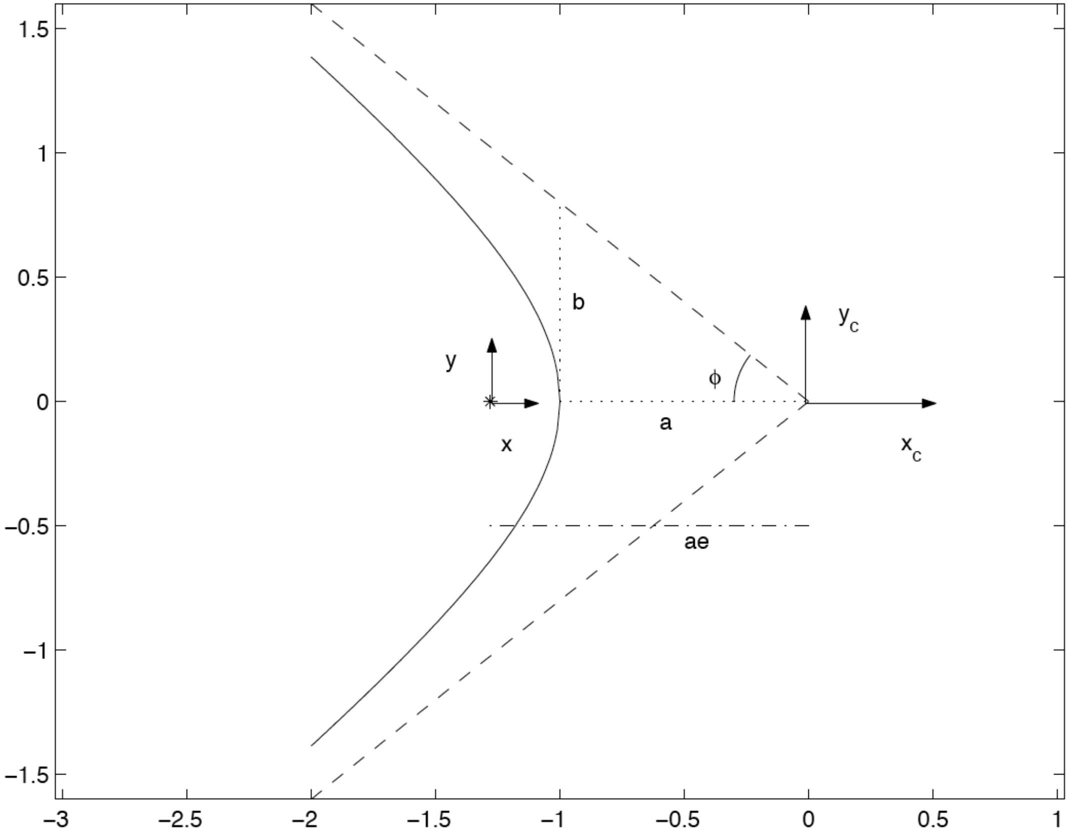
- *Circular speed:* $r \equiv a$

$$v_c = \sqrt{\frac{GM_{sun}}{r}}$$

- *Escape speed:* $a \to \infty$

$$v_e = \sqrt{\frac{2GM_{sun}}{r}} = \sqrt{2}v_c$$

The Hyperbola



$$\left(\frac{x_c}{a}\right)^2 - \left(\frac{y_c}{b}\right)^2 = 1$$

$$r(\nu) = \frac{p}{1 + e\cos\nu}$$

$$q = a(1 - e)$$

$$p = a(1 - e^2)$$

$$\cos\nu_\infty = -\frac{1}{e}$$

$$\mathcal{E} = -\frac{\mu}{2a}$$

Semi-major axis *negative* for hyperbolas!

Hyperbolic deflection

Impact parameter: *B* (minimum distance along the straight line)

Velocity at infinity: V_∞ *Angle of deflection*: $\psi = \pi - 2\phi$

$$\cos\phi = \cos(\pi - \nu_\infty) = -\cos\nu_\infty = \frac{1}{e}$$

$$B = |a| e \sin\phi = |a|\sqrt{e^2 - 1} = b$$

$$\mu = GM$$

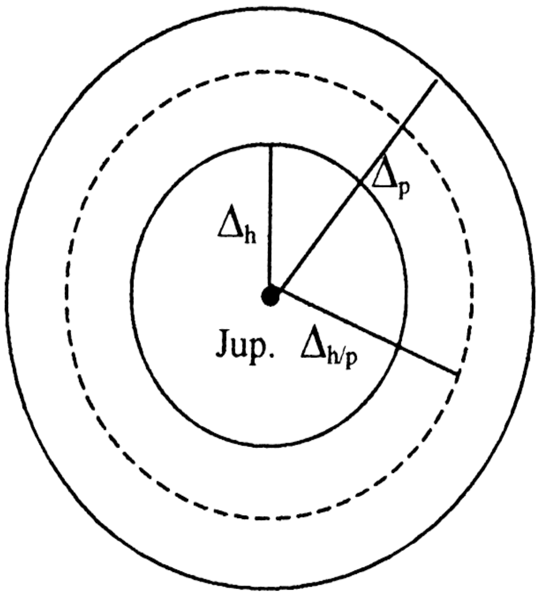
$$\tan\frac{\psi}{2} = \cot\phi = \frac{1}{\sqrt{e^2 - 1}} = \frac{\sqrt{\mu|a|}}{h}$$

$$h = V_\infty B \; ; \qquad V_\infty^2 = \frac{\mu}{|a|}$$

$$\Rightarrow \tan\frac{\psi}{2} = \frac{\mu}{V_\infty h} = \frac{\mu}{V_\infty^2 B}$$

Sphere of influence

- A massive body has a ***sphere of influence***, where its gravitational influence exceeds that of the Sun (e.g., the Hill sphere)
- This can be defined in terms of the *ratio of central to perturbing force* in the planetocentric or heliocentric frame

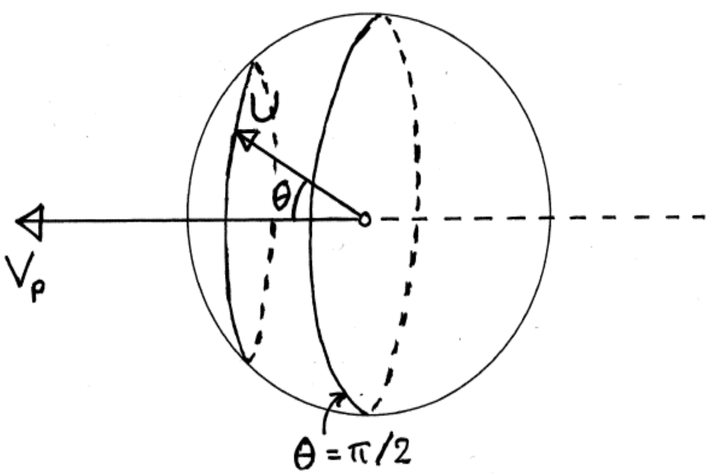


Close encounters

- Approximate treatment as ***hyperbolic deflections*** (scattering problem)
- The approach velocity *U* is conserved:

$$U^2 = 3 - T$$

- As the direction of the velocity vector is changed, the heliocentric motion can be either accelerated or decelerated



θ *controls the values of* E *and* H_z

Gravitational scattering

- *Protoplanet-planetesimal interactions* in the early Solar System (planetary migration due to exchange of energy and angular momentum)
- *Capture of comets* into short-period orbits
- *Gravity-assist manoeuvres* in space missions (ex. Grand Tour)

Gravitational focusing

The actual minimum distance is smaller than the impact parameter

$$q = |a|(e-1) \qquad B = |a|\sqrt{e^2 - 1}$$

$$\frac{q}{B} = \sqrt{\frac{e-1}{e+1}}$$

Cross-section for collision with a planet: impact parameter yielding a grazing collision

$$v_\infty \cdot B_g = \sqrt{v_\infty^2 + v_e^2} \cdot R_p$$

Planetary Perturbations (1)

- *Small departures from Keplerian motion, expressible as (mostly slow) changes of the orbital elements*

Formulating the equation of motion in the heliocentric frame:

- ***Direct perturbations***: caused by the acceleration of the test body due to the perturbing planet
- ***Indirect perturbations***: caused by the acceleration exerted by the perturbing planet on the Sun

Planetary Perturbations (2)

- ***Perturbing function:***

DIRECT INDIRECT

Pert. Acc. = $\nabla \boldsymbol{R}_1$

$$R_1 = GM_1 \left\{ \frac{1}{\left|\vec{r} - \vec{r}_1\right|} - \frac{\vec{r}_1 \cdot \vec{r}}{r_1^{\,3}} \right\}$$

- If M_1 and R_1 were zero, the orbital elements would be constant
- When M_1 and R_1 are small, the orbital elements will vary slowly
- Set up differential equations for the time derivatives of (a,e,i,ω,Ω,T), seek solutions that are valid over as long time as possible

Lagrange’s planetary equations

$$\frac{da}{dt} = \frac{2}{na}\frac{\partial \Re}{\partial \mathcal{X}}$$

$$\frac{de}{dt} = \frac{1}{na^2 e}\left((1-e^2)\frac{\partial \Re}{\partial \mathcal{X}} - \sqrt{1-e^2}\frac{\partial \Re}{\partial \omega}\right)$$

$$\frac{di}{dt} = \frac{1}{na^2\sqrt{1-e^2}}\left(\cot i\frac{\partial \Re}{\partial \omega} - \operatorname{cosec} i\frac{\partial \Re}{\partial \Omega}\right)$$

$$\frac{d\Omega}{dt} = \frac{1}{na^2\sqrt{1-e^2}\sin i}\frac{\partial \Re}{\partial i}$$

$$\frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial \Re}{\partial e} - \frac{\cot i}{na^2\sqrt{1-e^2}}\frac{\partial \Re}{\partial i}$$

$$\frac{d\mathcal{X}}{dt} = -\frac{1-e^2}{na^2 e}\frac{\partial \Re}{\partial e} - \frac{2}{na}\frac{\partial \Re}{\partial a},$$

Series development

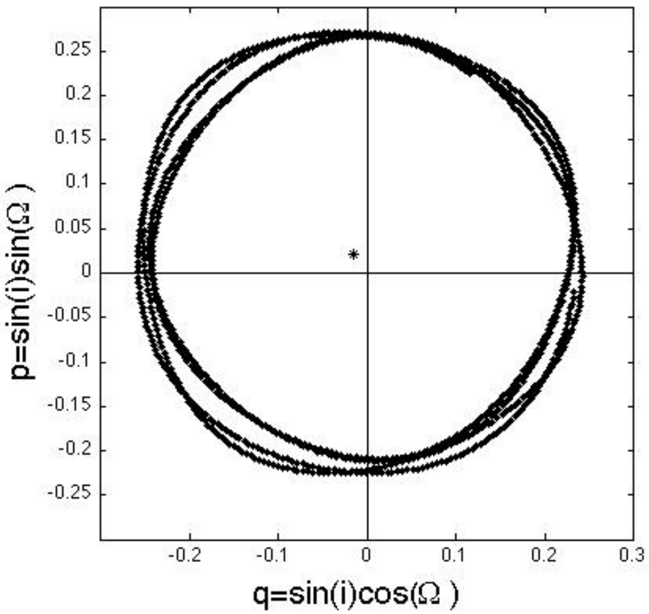
- Use successive approximations for solving the planetary equations, starting from a linear perturbation theory, computing the perturbing function from unperturbed orbits
- Express the coordinates, and thus the perturbing function, as trigonometric series in the angular elements (M,ϖ,Ω) with coefficients that are non-linear expressions in the non-angular elements (a,e,i)
- Truncate this expression at a certain order in the small quantities (e and i)

Action-angle variables

- By truncating the series developments, we necessarily get an integrable system of equations
- By transforming the variables, we can put this system on a canonical form, which means that the ***Hamiltonian*** (energy of the system) only depends on three of them (the action variables) but not on the other three (angle variables)
- *The integrals are then the action variables and the frequencies of the angle variables* (constants of motion)
- *The semi-major axis a is such a constant in the linear theory*, *and M is an angle variable with constant frequency n (the mean motion)*

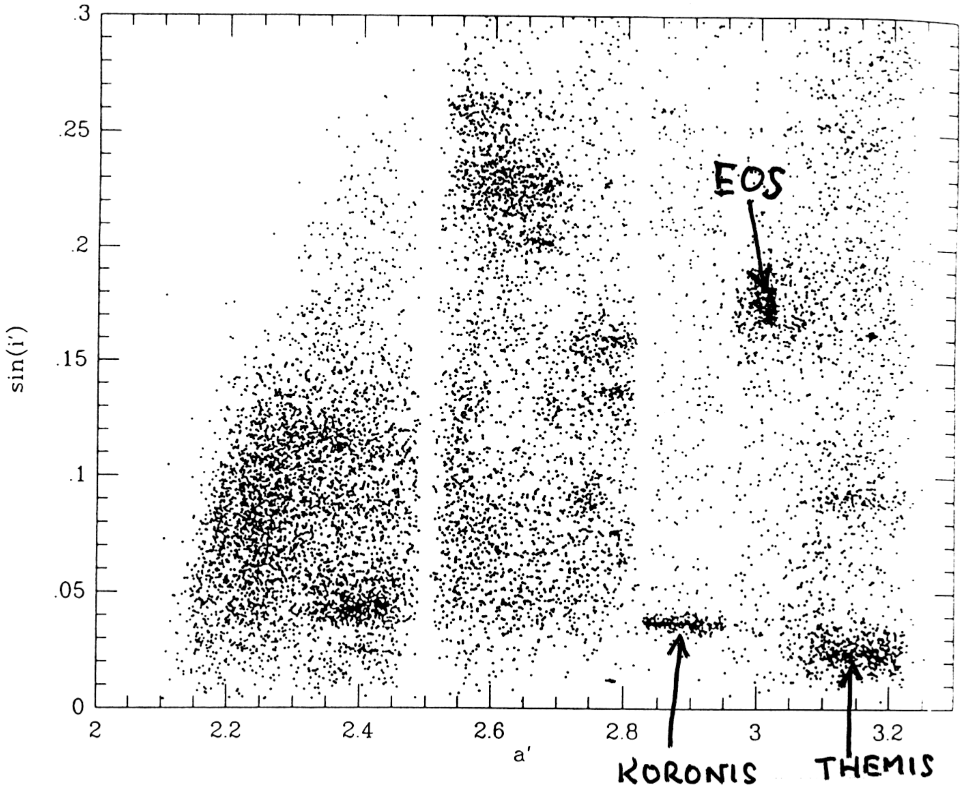
Proper Elements

- The linear theory also identifies the coupled variations of (e,ω) and (i,Ω) in terms of quasi-circular patterns in e.g. the ($e\cos\omega, e\sin\omega$) plane
- The offset of the center is the "forced eccentricity" and the radius of the circle is the "proper eccentricity"
- *The proper eccentricity and inclination are quasi-constants of motion over very long periods of time*



Example from a long-term integration of (11798) Davidsson

Asteroid Collisional Families



- *Hirayama families* (Hirayama 1918), most evident using proper elements: **Eos, Koronis, Themis**