Dynamics of planet formation
Centrifugal equilibrium

• Conservation of angular momentum in a contracting cloud ⇒ increase of angular velocity and rotational energy

• Rotational energy $U_{\text{rot}}$ increases faster than gravitational energy $U$ decreases

• Contraction perpendicular to the spin axis stops when the centrifugal force equals the force of gravity at the equator of the cloud

• $R_{\text{lim}} = R_c$ (centrifugal radius)

• Continued collapse along the spin axis ⇒ flattened disk with radius $R \sim R_c$
Rotation and Contraction

- Potential energy of a spherical, homogeneous cloud:
  \[ U = -\frac{3}{5} \frac{GM^2}{R} \]

- Moment of inertia:
  \[ I = \frac{2}{5} MR^2 \]

- Energy of rotation:
  \[ U_{rot} = \frac{1}{2} I \omega^2 = \frac{L^2}{2I} = \frac{5}{4} \frac{L^2}{MR^2} \]

- Centrifugal equilibrium condition:
  \[ R_c \omega^2 = \frac{GM}{R_c^2} \quad \therefore R_c = \frac{25L^2}{4GM^3} = \frac{25}{4GM} \times \left( \frac{L}{M} \right)^2 \]

Centrifugal radius
Estimating a rough value of $R_c$

- Use $L/M = \omega R^2$ from observations of molecular cloud cores
- $R=5000$ AU and $\omega=2 \times 10^{-14}$ s$^{-1}$ leads to $R_c \sim 25$ AU

- Observations of protoplanetary disks are consistent with this estimate
Protoplanetary disks

HH 30
“Herbig-Haro object”

“proplyds”
(Orion nebula)

General radii ~ 100 AU

HST pictures
Accretion disk

• Differentially rotating disk - *angular velocity decreases outward: SHEAR*

• The shear has a physical effect, if elements at different radial distances “stick” together

• For a gaseous disk the interaction can be described as a *VISCOSITY* (suppressing relative motion)
Shearing instability, turbulent flow

- **Autobahn example:**
  - Low traffic, peaceful drivers – *laminar flow*
  - High traffic, nervous drivers changing lanes – *turbulent flow*
Consequences of shear and viscosity

- The energy of relative motion is dissipated ⇒ HEATING
- Angular momentum is transported outward
- Global tendency for the disk to break up radially
- Material deprived of angular momentum collects at the center of the disk: ACCRETION
Accretion disk evolution

- Consider a *circular annulus* according to the picture
- The surface density is $\Sigma$
- $\Omega$ decreases with $r$
- In the absence of infall:

$$2\pi r \Delta r \frac{\partial \Sigma}{\partial t} = \frac{\partial (\Delta m)}{\partial t} = -\Delta (2\pi r \Sigma v_r)$$

Thus the *mass continuity equation*:

$$r \frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial r} (r \Sigma v_r) = 0$$
Accretion disk evolution, ctd

Conservation of angular momentum:

\[ 2\pi r \Delta r \frac{\partial}{\partial t} \left( \Sigma r^2 \Omega \right) = \frac{\partial (\Delta L)}{\partial t} = -\Delta \left( 2\pi r^3 \Sigma \Omega v_r \right) + \Delta G \]

where \( G \) is the \textit{torque} acting on the inner neighbour.

Thus the \textit{angular momentum continuity equation}:

\[ r \frac{\partial}{\partial t} \left( \Sigma r^2 \Omega \right) + \frac{\partial}{\partial r} \left( r^3 \Omega \Sigma v_r \right) = \frac{1}{2\pi} \frac{\partial G}{\partial r} \]

Shear velocity:

\[ A = r \frac{d\Omega}{dr} \]
Shear and viscosity

- Viscous tension: $p = \rho \nu A$
  where $\nu$ is viscosity

- Force acting over length $\ell$:
  $F = \ell \int p \, dz = \ell \cdot \nu \Sigma A$

- Total torque $G = rF$
  with $\ell = 2\pi r$:

  $G = r \cdot 2\pi r \cdot \nu \Sigma A = 2\pi r^2 \nu \Sigma A = 2\pi r^3 \nu \Sigma \frac{d\Omega}{dr}$
Global evolution

• Combining the equations, eliminating $v_r$, we get a diffusion equation for $\Sigma$:

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left\{ \frac{1}{d/dr(r^2\Omega)} \cdot \frac{\partial}{\partial r} \left[ r^3 \nu \Sigma \left( \frac{-d\Omega}{dr} \right) \right] \right\}$$

**Example:** a narrow, circular ring gets dispersed inward and outward over time. Most of the mass moves inward, and a small fraction of the mass carries the angular momentum outwards.
Magneto-rotational instability

- Suppose the gas disk is partially ionized and penetrated by a frozen-in magnetic field
- If a field line connects two gas parcels at somewhat different radial distance, their differential rotation will stretch the field line, and magnetic tension acts to keep them together $\Rightarrow$ instability against radial break-up

Initiates turbulence and transports angular momentum
Episodic accretion

- A "dead zone" may develop inside the disk, where neither thermal motion nor X-rays are able to ionize the gas enough for MRI.
- The dead zone would accumulate material from the outside and grow in mass.
- There may be bursts of accretion, when such clumps fall onto the protostar.
Planetary formation mechanisms

- Planetesimal Accretion (PA)
- Gas Disk Instability (GI)

*Terrestrial planets* were certainly formed by PA

For *giant planets*, PA is favoured

But the issue is still open!
What is a planetesimal?

• A **planetesimal** is a planetary building block large enough to cause some significant *gravitational attraction* on neighbouring objects (km-size or larger)

• A **planetary embryo** is formed from planetesimals by accumulating them into a larger object by local accretion

*The eventual planets are much fewer than the embryos*
Planetesimal formation

- Proposed mechanisms:
  - gravitational instability of a thin disk of solid bodies
  - grain accretion by collisions with sticking, triggered by gas-dust interactions
Gravitational instability

*Compare the derivation of the Jeans mass!*

- Consider a *disk in differential, Keplerian rotation* around a central object (Sun)
- *Perturb the mass distribution locally; find the response of the disk as a spiral wave*  
- The dispersion relation (frequency vs wave number) indicates *instability* (imaginary frequency) *for large wavelengths, if the velocity dispersion is small enough*  
- Hence an instability so that local density enhancements can grow ⇒ *planetesimals*
Grain growth

• Mechanisms for grain collisions:
  - *thermal* (Brownian) motions
  - *turbulent* gas motions (grains carried along by eddies)
  - gravitational *settling* into the midplane
  - *inward drift* due to solar gravity and gas drag
Gas drag

- Consider a *grain moving sub-sonically* through a gas with molecular mass $m_g$ and thermal speed $v_g$. The drag force acting on the grain is:

$$F_D = n_g A V \times C_D m_g v_g$$
Coupling time scale

• Solving the equation of motion

\[ M \frac{dV}{dt} = F_D \]

we get:

\[ V = V_o \exp \left\{ -C_D \frac{A}{M} n\mu u_{th} t \right\} = V_o \exp \left\{ -\frac{t}{\tau_c} \right\} \]

with

\[ \tau_c = \frac{M}{AC_D \rho_{gas} u_{th}} \quad \left( \rho_{gas} = n\mu \right) \]

• For a spherical grain with radius \( R_{gr} \) and density \( \rho_{gr} \), the gas-grain coupling time scale is roughly:

\[ \tau_c \approx \frac{\rho_{gr} R_{gr}}{\rho_{gas} u_{th}} \quad \text{(very short!)} \]
Grain settling to the midplane

- Grav. force on a grain of mass $M$:
  \[ F_G = M \Omega^2 r \quad \left( r^3 \Omega^2 = GM_{\text{sun}} \right) \]

- $z$ component:
  \[ F_z = M \Omega^2 z \]

- Gas drag force on a grain with radius $R_{gr}$ moving at vertical velocity $u$:
  \[ F_D = \left( \pi R_{gr}^2 n u \right) \times \left( C_D \mu u_{th} \right) \]

\[ \Omega = \text{local frequency of disk rotation} \]
\[ C_D \sim 1 = \text{gas drag coefficient} \]
\[ u_{th} = \text{random velocity of a gas molecule in the direction opposite to } u \]
Settling time scale

- With \( \rho_{\text{gas}} = n\mu = \) density of the gas disk and \( \rho_{\text{gr}} = \) density of the grain, force balance yields:
  \[
  \rho_{\text{gas}} u_{\text{th}} u_{eq} = \frac{4}{3} R_{\text{gr}} \rho_{\text{gr}} \Omega^2 z
  \]

- With \( u_{eq} = dz/dt \), we get:
  \[
  z = z_0 \exp(-t/\tau_s)
  \]
  introducing the sedimnetation time scale:
  \[
  \tau_s \approx \frac{z}{u_{eq}} = \frac{3\rho_{\text{gas}} u_{\text{th}}}{4 R_{\text{gr}} \rho_{\text{gr}} \Omega^2}
  \]

~1000 yr for cm-sized grains at Jupiter’s orbit
Radial drift

- The gas circulates at sub-Keplerian speed being pressure supported, but the grain is not.
- When the grain is small and coupled to the gas, it is dragged toward the Sun by moving too slowly.
- When the grain is larger and decoupled from the gas, it moves at Keplerian speed and feels a gas drag. Thus it loses speed and spirals toward the Sun.

This radial drift causes the grain to sweep up smaller ones.
Drift time scale

- Orbital angular momentum: \( L = M \sqrt{GM_{\text{sun}} r} \)
- Drag torque: \( \tau = r F_D = r \rho_{\text{gas}} A u^2 C_D \)
- Decay time scale:
  \[
  T_d = \frac{L}{\tau} \approx R_{gr} \frac{\rho_{gr}}{\rho_{\text{gas}}} u^{-2} \sqrt{\frac{GM_{\text{sun}}}{r}}
  \]
- Taking \( u \) (the difference between Kepler and pressure-supported speeds) as 1\% of the circular speed:
  \[
  T_d \approx R_{gr} \frac{\rho_{gr}}{\rho_{\text{gas}}} \cdot 10^4 \sqrt{\frac{r}{GM_{\text{sun}}}}
  \]

A few hundred years for a 1-meter boulder **Problem!**
Planetary accretion

- **Collective growth** of size distribution and relative velocity distribution in an enormous population of initially roughly equal-sized planetesimals

- **Runaway growth**: the largest body grows much more rapidly than the rest of the population ⇒ quick formation of planet embryos, moving in concentric orbits separated by a few Hill radii from each other

*Phenomena caused by the gravity of planetesimals: Stirring and Focussing*
Close encounters

• The number of planetesimals per unit area is so large that “congestion” occurs ⇒ frequent close encounters

• Gravity causes hyperbolic deflections with $\psi = \psi(b, v_o, m)$; this increases the orbital eccentricities
Stirring and focussing

• The typical eccentricity/inclination obtained by hyperbolic deflections (“gravitational stirring”) determines the average relative velocity $v_o$ at close encounters

• If $m_1$ is the largest planetesimal mass ($r_1$ is the radius), we have:

$$v_o^2 = \frac{Gm_1}{\theta r_1} = \frac{1}{2\theta} \cdot v_{e,1}^2$$

(V.I. Safronov)

where $v_e$ is the escape speed and $\theta \sim 2-5$

• From angular momentum conservation we get:

$$v_o \cdot r_c = \sqrt{v_o^2 + v_e^2} \cdot r$$

where $r_c > r$ is the maximum impact parameter for collision (“gravitational focussing”), and we get the cross-section:

$$A_c = \pi r^2 \left(1 + \frac{v_e^2}{v_o^2}\right) \leftarrow \text{gravitational focussing}$$