Stellar Dynamics

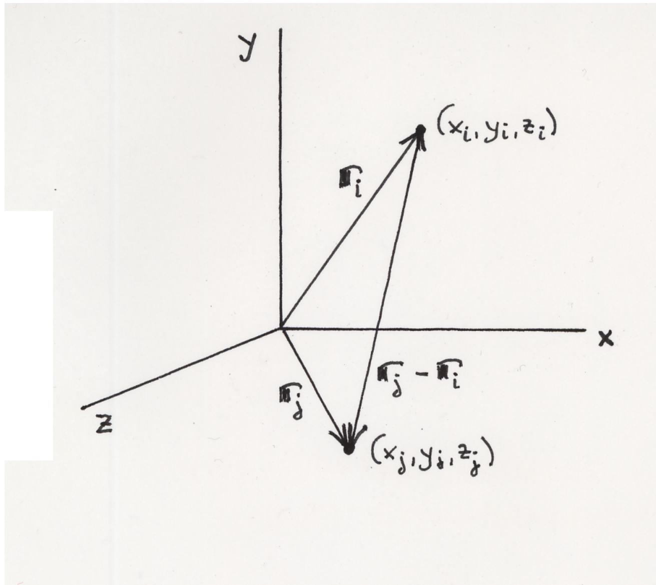
Part I

The N-Body Problem

$$r_{ij} = |\mathbf{r}_j - \mathbf{r}_i| = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2}$$

- Forces acting within the system:

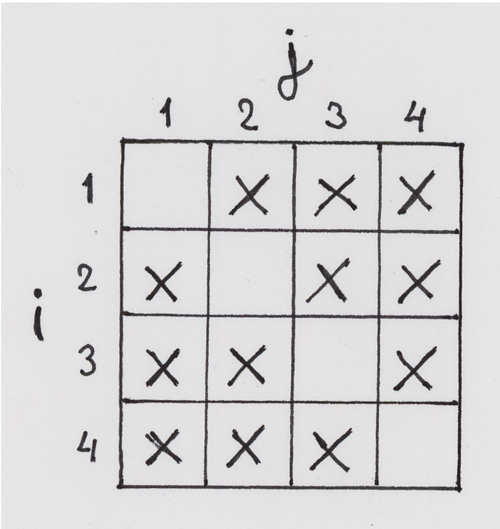
$$m_i \ddot{\mathbf{r}}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G m_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \qquad (i = 1, \ldots, N)$$



Dynamical system of order 6N

Numerical integrations vs analytic solutions

Center of Mass integrals



- The sum of the equations:

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G m_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \qquad (i = 1, \ldots, N)$$

- The terms cancel out in pairs:

$$\sum_{i=1}^{N} m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = \mathbf{0}$$

- Center of mass position:

$$\mathbf{r}_g = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\mathbf{a}t + \mathbf{b}}{M}$$

Eliminate the common motion, solve for the internal motions

Angular momentum integrals

- Sum of cross products:

$$\sum_{i=1}^{N} m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G m_i m_j \mathbf{r}_i \times (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \qquad (i = 1, \ldots, N)$$

- Cancellation of terms: $\sum_{i=1}^{N} m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = \mathbf{0}$

- Integration: $\sum_{i=1}^{N} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \mathbf{c}$

Conservation of the total angular momentum vector

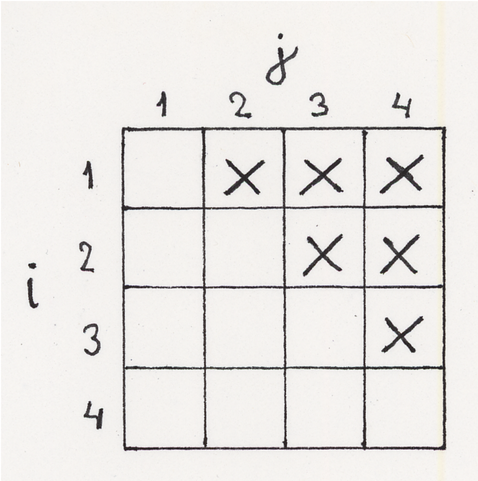
Characteristic direction (axis of symmetry)

Energy integral

- Pairwise potential energy: $$\Omega_{ij} = -\frac{Gm_i m_j}{r_{ij}}$$
- Total potential energy:

$$\Omega = -\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{Gm_i m_j}{r_{ij}}$$

$$m_i \ddot{\mathbf{r}}_i = -\nabla_i \Omega$$



- Multiply by velocity and sum:

$$\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i + \sum_{i=1}^{N} \dot{\mathbf{r}}_i \cdot \nabla_i \Omega = 0$$

- Integration:

$$\boxed{T + \Omega = E}$$

Conservation of the total energy

Treatments of the problem

- N = 2: **analytic solution** (Kepler's laws)
  two-body problem
- N = a few: "experiments" using
  numerical simulations
  few-body problem
- N ~ 100 or more: **statistical mechanics**
  many-body problem

Lagrange's identity

- Moment of inertia about the origin:

$$J = \sum_{i=1}^{N} m_i \mathbf{r}_i^2$$

- Second time derivative:

$$\ddot{J} = 2\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i^2 + 2\sum_{i=1}^{N} m_i \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i$$

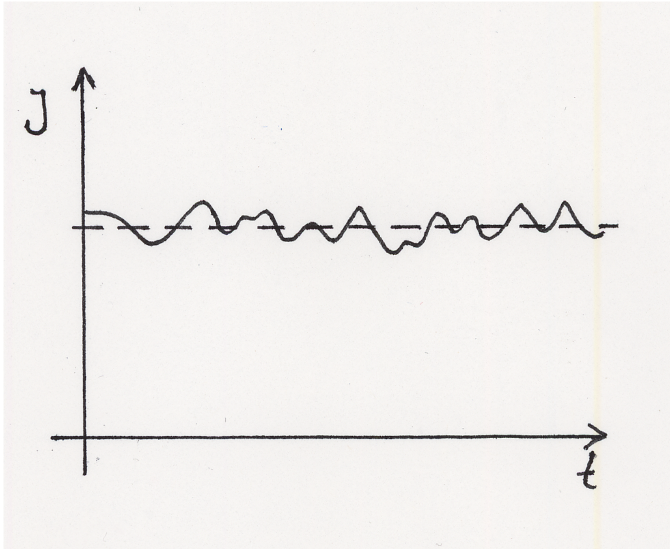
- Simplification:

$$\boxed{\ddot{J} = 4T + 2\Omega}$$

Lagrange's identity

Virial equilibrium

- Stellar systems in general are in ***steady state*** in spite of stellar motions
- Thus, *J = constant*, apart from statistical fluctuations, and as long as the system does not evolve dynamically
- On the average, we get from Lagrange's identity:



$$4\overline{T} + 2\overline{\Omega} = 0$$

Virial theorem

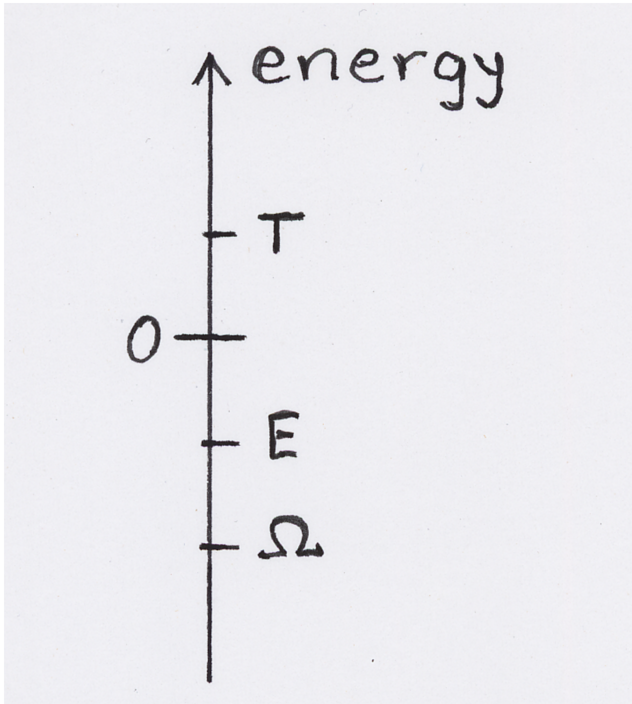
- For many-body systems like star clusters or galaxies, the statistical fluctuations are negligible
- Using the energy integral:

$$\boxed{T = -E \qquad ; \qquad \Omega = 2E}$$

and

$$\boxed{2T + \Omega = 0}$$

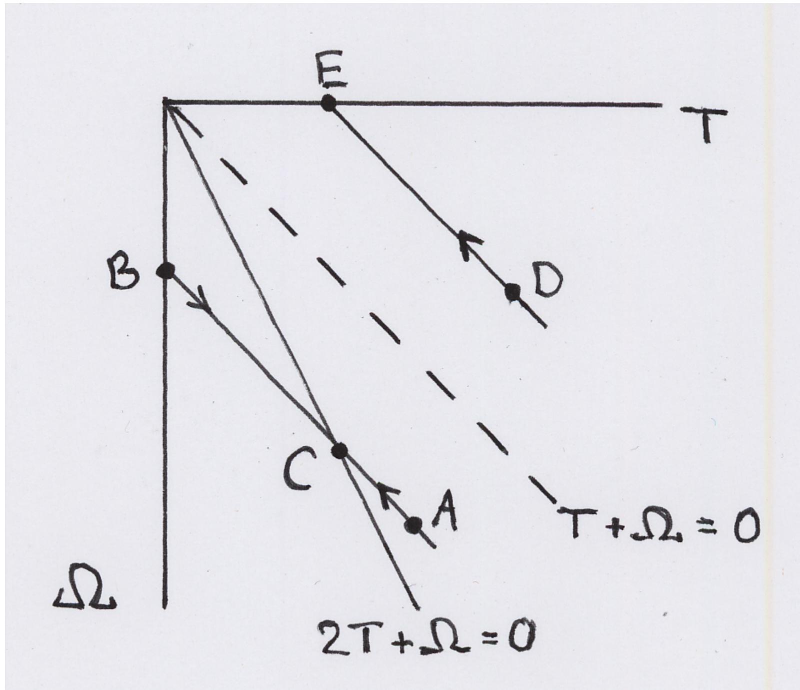
Virial theorem



Energy relations for steady-state systems

Virialization

- A system that is out of virial equilibrium tends to evolve toward it, while **conserving total energy**
- This is possible only if the energy is negative
- Virialization involves *damped oscillations* due to overshoot



Practical applications

- Approximating the terms in the virial theorem, and neglecting correlations:

$$\langle v^2 \rangle \approx \frac{GM}{2\langle r \rangle}$$

- For star clusters: mass and size yield the velocity dispersion ~ 1 km/s
- For galaxy clusters: size and velocity dispersion yield mass >> “observed” mass

Missing mass: dark matter

Statistical Description

- the Distribution function
- the Liouville equation
- Close encounters

Distribution function

- **Phase space** coordinates:

$$x_i, y_i, z_i, u_i, v_i, w_i, m_i \quad i = 1, \ldots, N$$

- $\Psi(x, y, z, u, v, w, m, t)$ is a continuous, smooth function approximating the density of representative points in phase space

$$\int \Psi \, d\tau = N$$

Density and potential

- *Mass density* in physical space:

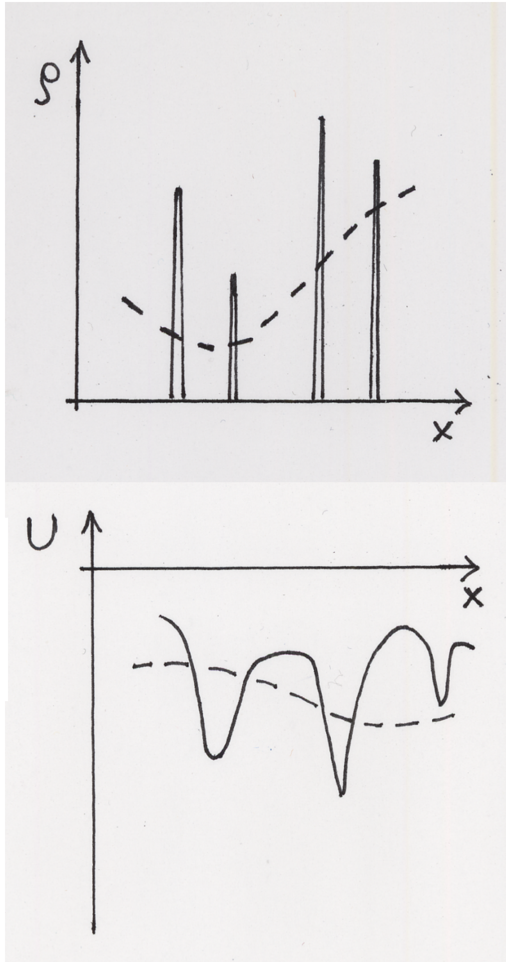
$$\rho(x,y,z,t) = \int_0^{\infty} m\,dm \int\int\int_{-\infty}^{+\infty} \Psi\, du\,dv\,dw$$

- *Gravitational potential*:

$$U(x_1,y_1,z_1,t) = -G\int\int\int_{-\infty}^{+\infty} \frac{\rho(x_2,y_2,z_2,t)\,dx_2dy_2dz_2}{r_{12}}$$

- ***Poisson equation***:

$$4\pi G\rho = \nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$



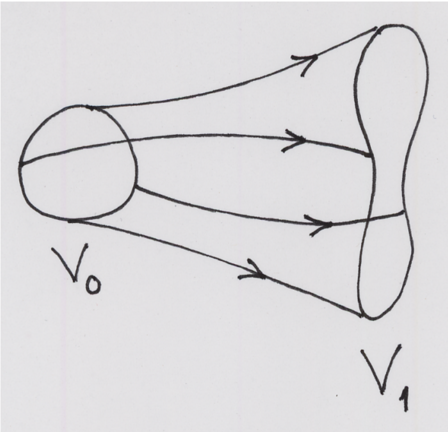
We neglect the potential wells around the stars!

Liouville theorem

Phase space velocity vector:

$$\left.\begin{aligned}\dot{x} &= u &= f_x\\ \dot{y} &= v &= f_y\\ \dot{z} &= w &= f_z\\ \dot{u} &= -\partial U/\partial x &= f_u\\ \dot{v} &= -\partial U/\partial y &= f_v\\ \dot{w} &= -\partial U/\partial z &= f_w\\ \dot{m} &= 0 &= f_m\end{aligned}\right\}\mathrm{f}$$

f *is a unique function of position – like fluid flow*



$$\boxed{\mathrm{div}\ \mathrm{f} = 0}$$

Phase space volumes are conserved

The distribution function is constant along phase space trajectories

Liouville equation

$$\frac{\partial \Psi}{\partial t} + \text{div}\,(\mathbf{f}\Psi) = 0$$

$$\frac{\partial \Psi}{\partial t} + \mathbf{f} \cdot \boldsymbol{\nabla}\Psi + \Psi\,\text{div}\,\mathbf{f} = 0$$

$$\boxed{\frac{\partial \Psi}{\partial t} + u\frac{\partial \Psi}{\partial x} + v\frac{\partial \Psi}{\partial y} + w\frac{\partial \Psi}{\partial z} - \frac{\partial U}{\partial x}\frac{\partial \Psi}{\partial u} - \frac{\partial U}{\partial y}\frac{\partial \Psi}{\partial v} - \frac{\partial U}{\partial z}\frac{\partial \Psi}{\partial w} = 0}$$

"Liouville equation" or
"collision-free Boltzmann equation"

Close encounters

- Real stellar systems are not entirely smooth and continuous; this depends on the number of stars
- The importance of local attractions (potential wells) decreases with the number of stars
- The effect of close encounters is to deflect the motions of stars ("hyperbolic deflections")

Stars will "jump" between different phase space positions and trajectories $\Rightarrow$ a "collisional term" enters into the Boltzmann equation

Collision-free systems

- Dynamical mixing
- The Jeans theorem
- Spherically symmetric systems
- Plane-parallel geometry

The mixing time scale

Time scale of orbital motion: ***Crossing time***

$$t_c = \frac{\langle r \rangle}{\langle v \rangle}$$

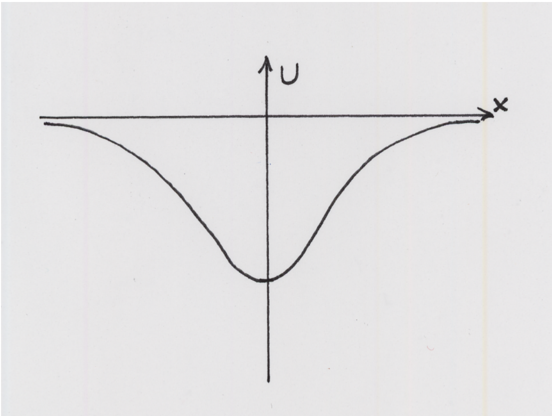
Use the virial theorem to substitute <v>:

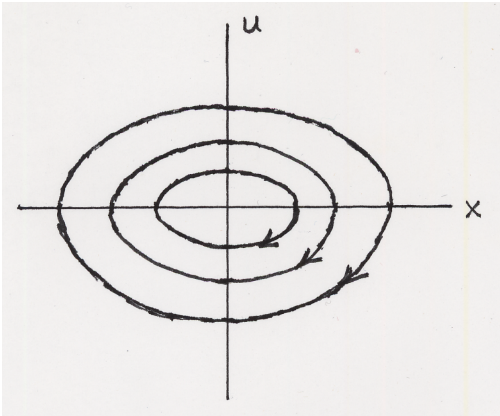
$$t_c = \sqrt{\frac{2\langle r \rangle^3}{GM}}$$

This is ~ 1 Myr for stellar clusters, much shorter than their ages, except for the youngest ones

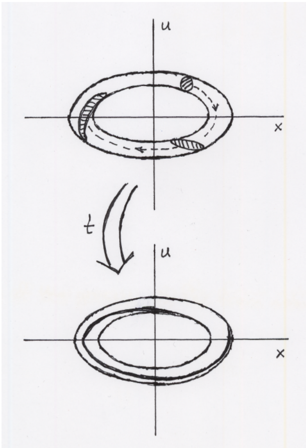
Phase space mixing

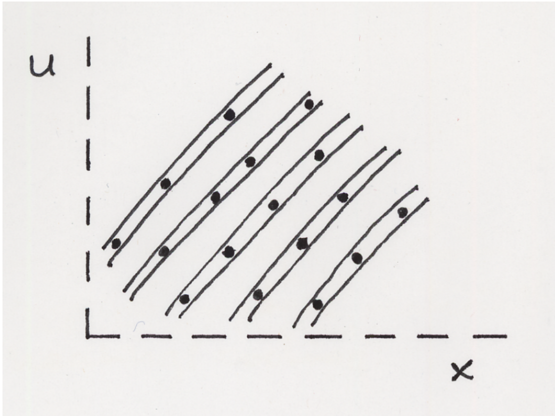
Example: one-dimensional system with attractive force





Evolution of a parcel of stars:





Elimination of the mass

Reduced distribution function: $\varphi = \int_0^\infty \Psi m \, dm$

(mass density in reduced 6-dimensional phase space)

Density equation: $\rho = \int\int\int_{-\infty}^{+\infty} \varphi \, du dv dw$

The Poisson equation is unchanged, and the Liouville equation looks the same with φ instead of Ψ

This is possible because we neglected the collision term

Integrals of motion

- An integral is a function $I(x,y,z,u,v,w,t)$ that remains constant along any trajectory
- Several integrals are independent, if there is no relation $g(I_1,I_2,\ldots,I_n)=0$ between them
- A sixth-order system has six independent integrals; all integrals are functions of these
- *But the distribution function is conserved along all trajectories!*

Jeans theorem

- The distribution function is an integral of motion; hence…
- ***The general solution of the Liouville equation is:***

$$\varphi = f(I_1, I_2, \ldots, I_6)$$

where the I's are six independent integrals of motion and f is an arbitrary function

Stationary systems

- For a system in steady state, φ cannot depend on time, so only time-independent integrals may feature in *f*
- Five of the six integrals can always be chosen time-independent ("conservative"), while the sixth may depend on time; hence
- ***In a stationary system the general form of the distribution function is***

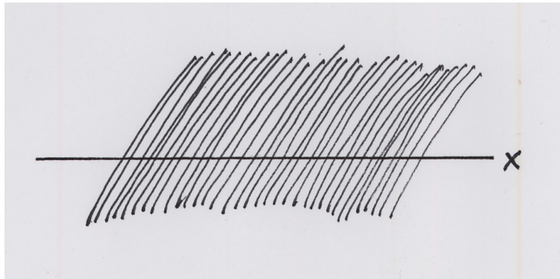
$$\varphi = f(I_1, I_2, \ldots, I_5)$$

where the I's are five independent, conservative integrals

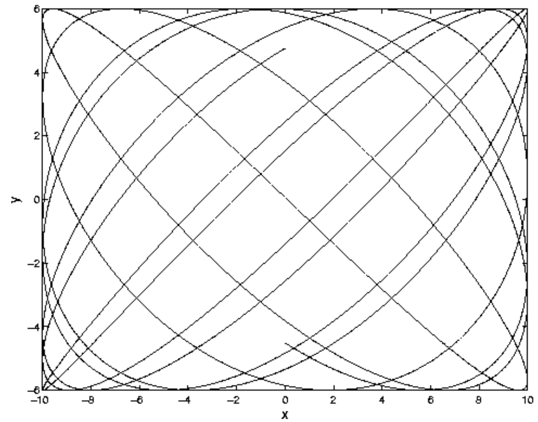
Non-Isolating integrals

- Simple example: 2D system with potential

$$U = \frac{1}{2}\left(a^2x^2 + b^2y^2\right)$$



- Solution: periodic oscillations in *x* and *y*
- One conservative integral is expressed by arctan functions and has values practically everywhere: ***"non-isolating"***
- For any value, the orbit comes arbitrarily close to any point (*x*,*y*)



Lissajous figure

Jeans theorem, final version

- Each conservative integral constrains the motion in 6D phase space to a 5D *"hyperplane"*
- But the hyperplanes of non-isolating integrals do not constrain the motion since they fill up the phase space everywhere
- The distribution function cannot depend on them; hence…
- ***In a stationary state, the distribution function is***

$$\varphi = f(I_1, I_2, \ldots, I_\gamma)$$

where the I's are independent, conservative, isolating integrals

The Energy Integral

- But only one such integral is known in the general case (i.e., without imposing symmetry properties on the system)
- The total energy of motion of a star per unit mass:

$$\mathcal{E} = \frac{1}{2}\left(u^2 + v^2 + w^2\right) + U(x, y, z)$$

- What about the four remaining conservative integrals? Are they all non-isolating? (*ergodic hypothesis*)
- No, but other isolating integrals are known only for special, symmetric systems