Stellar Dynamics

Part II

Spherically symmetric systems

M 22

- ***Globular clusters***
- Nearly spherically symmetric mass distribution
- Not necessarily collision-free!

Integrals

Spherically symmetric potential ⇒ central force field

$$\mathbf{A} = \mathbf{r} \times \dot{\mathbf{r}}$$

$$\frac{d\mathbf{A}}{dt} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}$$

Four independent, conservative, isolating integrals

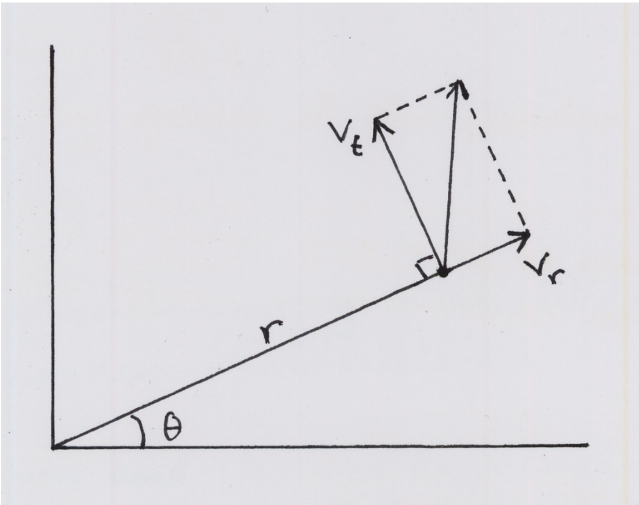
$$\varphi = f(\mathcal{E}, A_x, A_y, A_z)$$

$$A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\varphi = f(\mathcal{E}, A)$$

Orbits

Planar orbits $\perp$ **A**



- Velocity components:

$$\text{radial velocity} \qquad v_r = \dot{r}$$
$$\text{transverse velocity} \qquad v_t = r\dot{\theta}$$

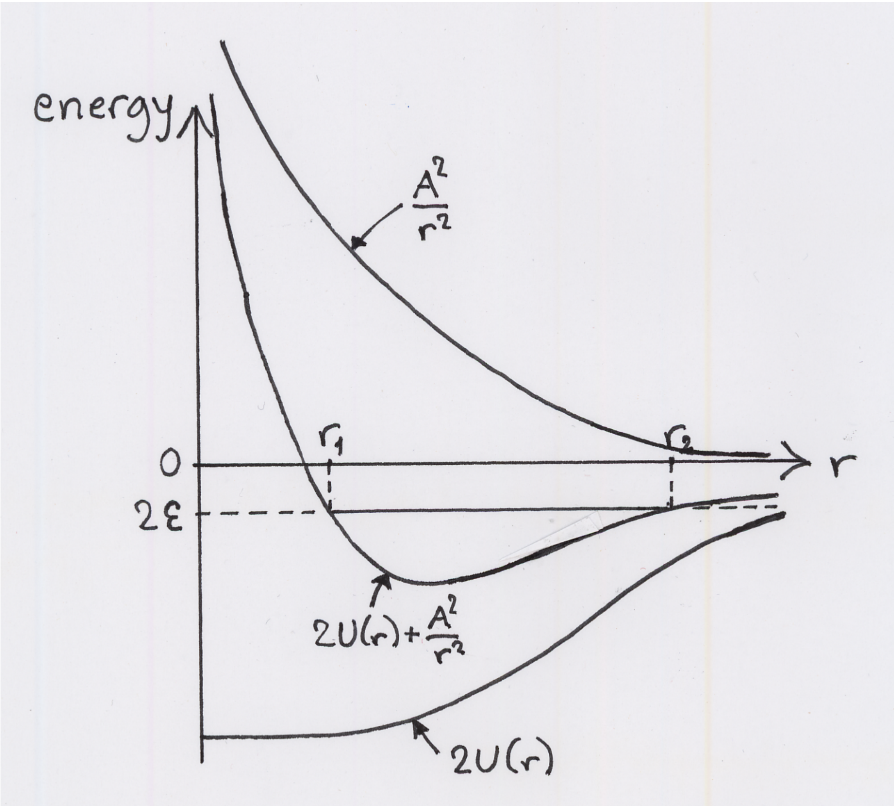
- Energy & Angular Momentum integrals:

$$\mathcal{E} = U(r) + \frac{1}{2}\left(v_r^2 + v_t^2\right) = U(r) + \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$A = r v_t = r^2 \dot{\theta}$$

- Solution: $\begin{cases} \dot{\theta} = A/r^2 \\ \dot{r} = \pm\sqrt{2\mathcal{E} - 2U(r) - A^2/r^2} \end{cases}$

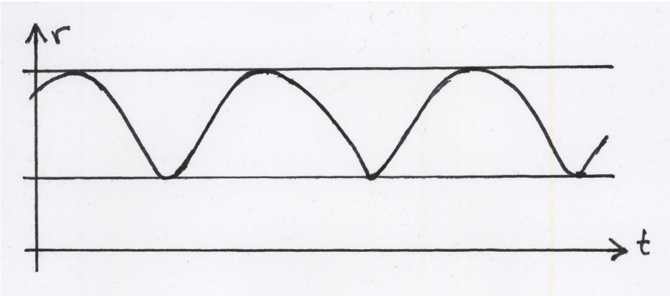
General appearance

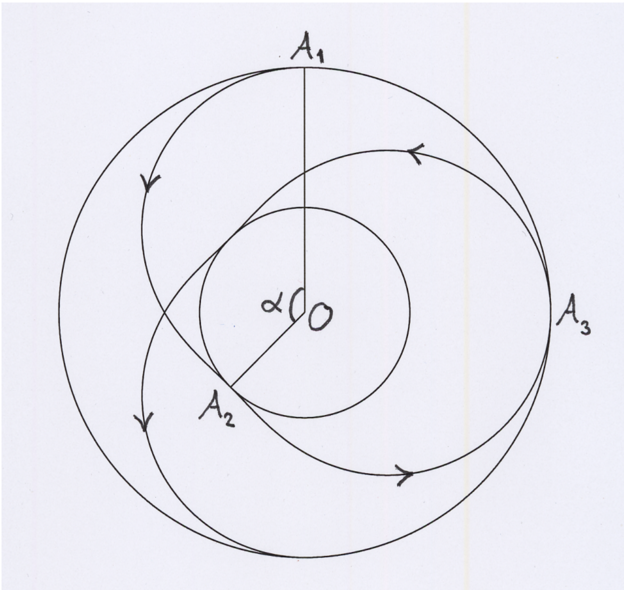


$$2U(r) + \frac{A^2}{r^2} \leq 2\mathcal{E}$$

- Angular momentum prevents $r \to 0$
- Energy may prevent $r \to \infty$

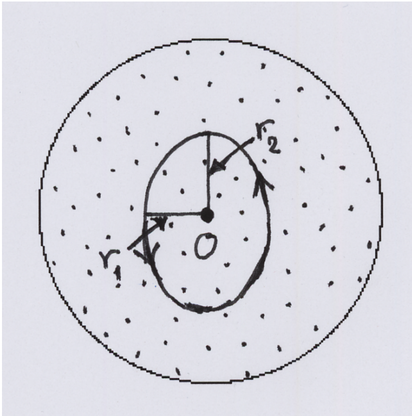
Rosette orbits



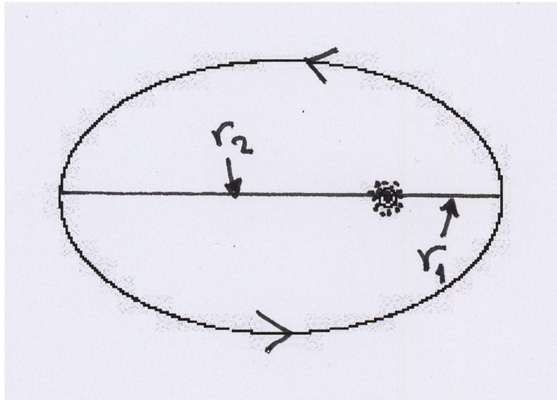


- *r* oscillates regularly between maxima and minima
- Combining this with a steady rotation in θ yields in general a rosette orbit that does not close upon itself
- Non-isolating fifth integral!

Extreme cases



Homogeneous system:
$\alpha = \pi/2$



Point-mass system:
$\alpha = \pi$

- In real stellar clusters, the homogeneous case is an approximation for the central parts and the point-mass case for the outskirts

Plummer model

Ansatz: $$\boxed{\varphi = \begin{cases} a(-\mathcal{E})^{7/2}; & \mathcal{E} < 0 \\ 0 \quad ; & \mathcal{E} \geq 0 \end{cases}}$$ (independent of *A*)

$$\varphi = g(U, V) = a(-U - V^2/2)^{7/2}$$

$$\rho = 4\pi \int_0^{V_\ell} a \left(-U - \frac{V^2}{2}\right)^{7/2} V^2 \, dV \qquad \rho = 4\pi a(-U)^5 \int_0^{\sqrt{2}} \left(1 - \frac{\tau^2}{2}\right)^{7/2} \tau^2 \, d\tau$$

Poisson equation: $$\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} = \begin{cases} 4\pi G C(-U)^5; & U < 0 \\ 0 \quad ; & U \geq 0 \end{cases}$$

Solution: $$U = \frac{U_o}{(1 + r^2/r_o^2)^{1/2}} \qquad \boxed{\rho = \frac{\rho_o}{(1 + r^2/r_o^2)^{5/2}}}$$ (polytrope)

Other models

For collisionally relaxed systems, one may use the Boltzmann energy distribution of kinetic gas theory:

$$\varphi \propto e^{-b\mathrm{E}}$$

However, this has to be cut at energies too large to exist in real clusters (*e.g.*, positive energies)

King models:

$$\boxed{\varphi = a\left\{e^{b(\mathcal{E}_t-\mathcal{E})} - 1\right\}; \quad \mathcal{E} < \mathcal{E}_t}$$

E_t is the "tidal" cutoff energy set by the tidal perturbations of the Galaxy

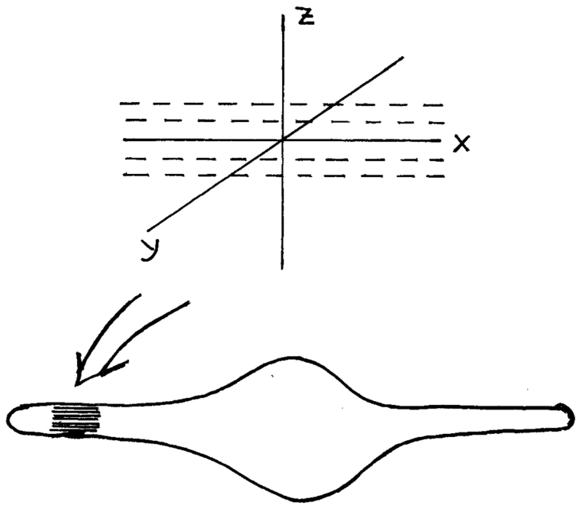
Eddington model:

$$\boxed{\varphi = a \cdot e^{-b\mathcal{E} - cA^2}}$$

Ellipsoidal velocity distribution stretched in the radial direction

Plane-parallel systems

NGC 4565



- φ, ρ, *U* depend only on the *z* coordinate
- This approximation can be used for disk galaxies, *e.g.*, the Galactic disk in the solar neighbourhood

Integrals

Integrate the distribution function over u and v:

$$\varphi_1(z,w) = \int\int \varphi\, du\, dv$$

$$\rho = \int \varphi_1\, dw$$

$$\nabla^2 U = \frac{\partial^2 U}{\partial z^2} = 4\pi G\rho \qquad\qquad w\frac{\partial \varphi_1}{\partial z} - \frac{\partial U}{\partial z}\frac{\partial \varphi_1}{\partial w} = 0$$

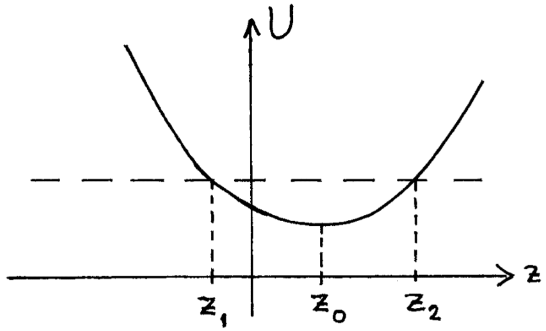
2-dimensional phase space; two independent integrals, whereof one conservative – the ***energy!***

Thus the general solution: $\varphi_1 = f(\mathcal{E}) \qquad \mathcal{E} = U(z) + \frac{w^2}{2}$

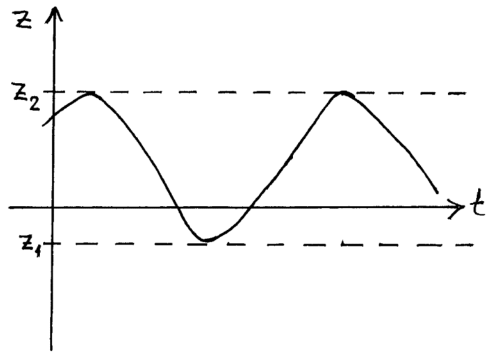
Orbits

- In x and y, the motion is uniform and rectilinear

$$\dot{z} = \pm\sqrt{2\mathcal{E} - 2U(z)}$$



- $U(z)$ is a concave function, corresponding to a restoring force toward $z=z_o$
- *The z motion is a periodic oscillation around $z=z_o$ with constant extrema z_1 and z_2*



An isothermal model

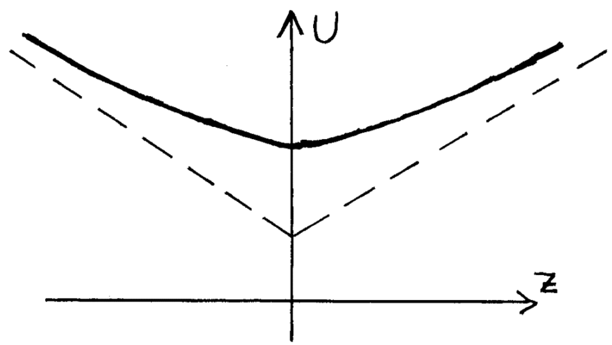
$$\varphi_1 = a \cdot e^{-bU} \cdot e^{-bw^2/2}$$

Relation between density and potential:

$$\rho = a\, e^{-bU} \int_{-\infty}^{+\infty} e^{-bw^2/2}\, dw = a\sqrt{\frac{2\pi}{b}} e^{-bU}$$

Poisson equation: $$\frac{\partial^2 U}{\partial z^2} = 4\pi G a \sqrt{\frac{2\pi}{b}} e^{-bU}$$

Solution: $$U = U_o + \frac{2}{b} \ln \coth \frac{z - z_o}{h}$$



This model only works for small values of |z|. The real disk potential flattens out at large distances.

Modelling the local disk

- Use a *tracer population*, e.g., K giants (relatively numerous and luminous stars)
- Observe the local velocity distribution $\varphi_{1K}(0,w) \Rightarrow f_K(E)$

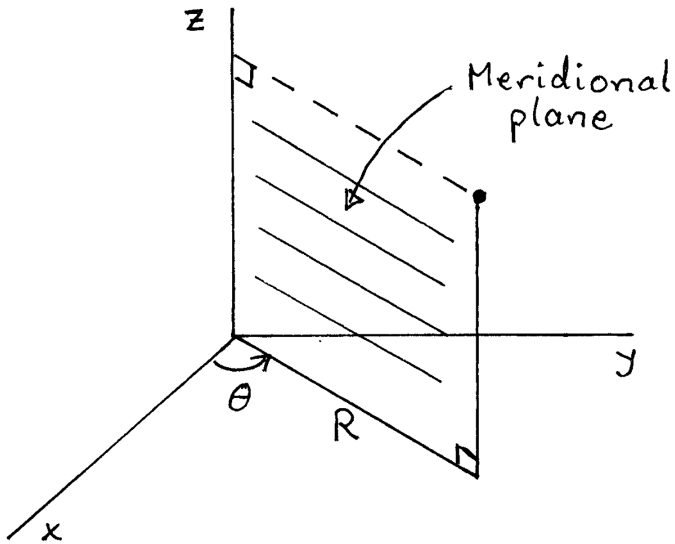
$$\rho_K(z) = \int f_K\left[U + \frac{w^2}{2}\right] dw = h_K(U)$$

- Observe $\rho_K(z) \Rightarrow U(z)$; use the Poisson equation to derive $\rho(z)$, including the local density ρ_o

From analysis of Hipparcos data, Holmberg & Flynn (2000) derive: $\rho_o = 0.10\ M_\odot/\mathrm{pc}^3$

Axial symmetry

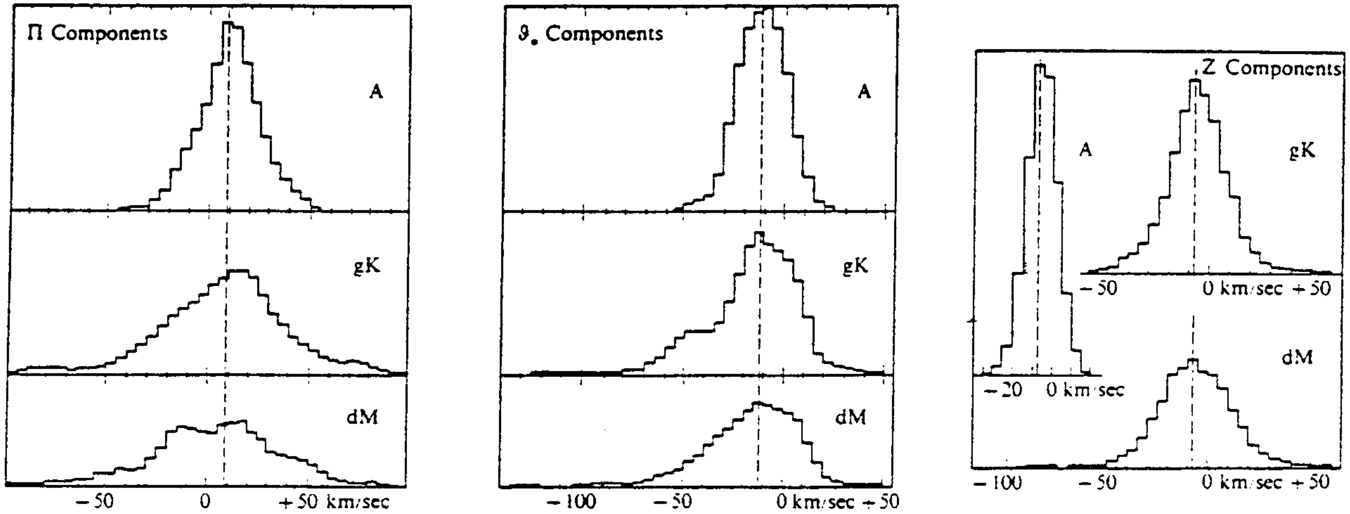
- Use *cylindrical coordinates*: (R,θ,z)
- *Meridional plane*: passes through the point and the z axis
- The force vector is in the meridional plane but not in general toward the origin; thus there may be a torque but with no z component



The z component of the angular momentum is conserved, so there are two integrals: E ***and*** A_z

No further integrals are known in the general case

Velocity ellipsoid



- If all three remaining conservative integrals are non-isolating, we have $\varphi = \varphi(E, A_z)$:

$$\varphi = f\left[\frac{1}{2}\left(u^2 + v^2 + w^2\right) + U\left(R_o, 0\right), R_o\, v\right]$$

- Thus the radial and vertical velocity components would have equal distributions, but this is not observed

Third Integral problem

- The non-equality of the axes of the velocity ellipsoid is one clear indication that there is a third, isolating integral
- Expressions for this have been found in special cases of simplified potentials
- Numerical exploration of a different simple potential by Hénon & Heiles (1964) led to the discovery of an *intricate mixture of integrable and ergodic motion* - thus a complex nature of the third integral
- This was one of the first demonstrations of *chaotic behaviour of dynamical systems*

Circular orbits

Circular velocity

linear

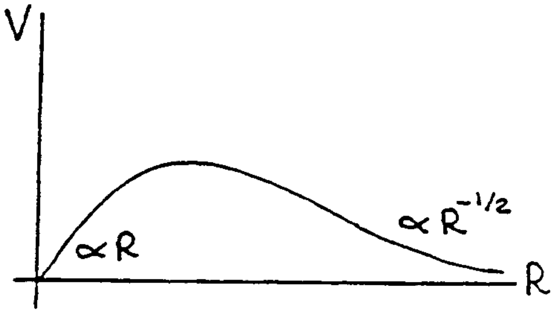
$$V^2 = R\frac{\partial U}{\partial R}$$

angular

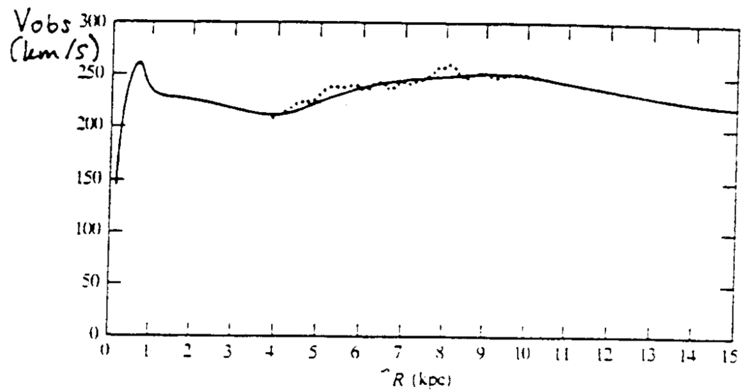
$$\omega^2 = \frac{1}{R}\frac{\partial U}{\partial R}$$

Rotation curve

theoretical

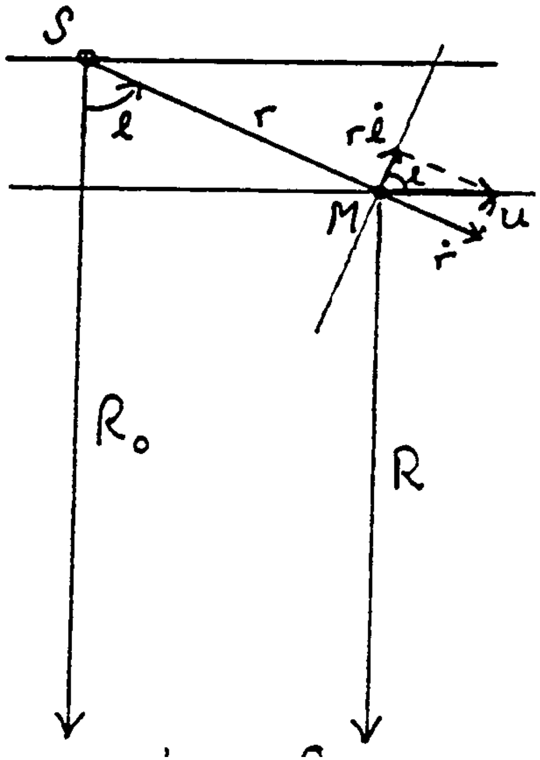


observed



Galactic halo

Differential rotation



$$\omega - \omega_o \simeq \left(\frac{d\omega}{dR}\right)_o \cdot (R - R_o)$$

$$u = R\left(\frac{d\omega}{dR}\right)_o \cdot (R - R_o)$$

$$\dot{r} = u \sin \ell$$

$$r\dot{\ell} = u \cos \ell$$

$$R - R_o \simeq -r \cos \ell$$

$$\dot{r} = -R_o \left(\frac{d\omega}{dR}\right)_o r \sin \ell \cos \ell$$

$$\dot{\ell} = -R_o \left(\frac{d\omega}{dR}\right)_o \cos^2 \ell$$

Oort's constants A and B

$$A = -\frac{1}{2} R_o \left(\frac{d\omega}{dR} \right)_o$$

$$B = A - \omega_o$$

Definitions

Observed velocities:

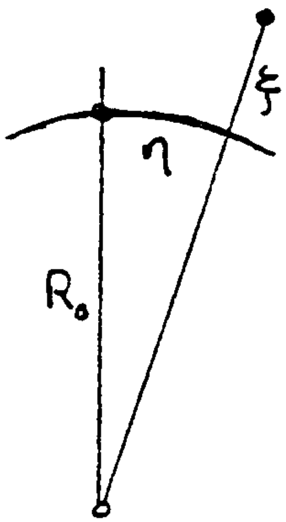
radial velocity

$$\dot{r} = A\, r\, \sin 2\ell$$

proper motion

$$\mu = \dot{\ell} - \omega_o = A \cos 2\ell + B$$

Nearly circular orbits



circular	***nearly circular***
$R = R_o$	$R = R_o + \xi$
$\theta = \omega_o t$	$\theta = \omega_o t + \dfrac{\eta}{R_o}$
$z = 0$	$z = z$

Find the time variations of ξ, η and z

Epicyclic motion

Equations of motion:

$$\ddot{R} - R\dot{\theta}^2 = -\frac{\partial U}{\partial R}$$

$$2\dot{R}\dot{\theta} + R\ddot{\theta} = 0$$

$$\ddot{z} = -\frac{\partial U}{\partial z}$$

Linearize in the small quantities:

$$\ddot{\xi} - 2\omega_o\dot{\eta}\ \omega_o^2\xi = -\frac{\partial^2 U}{\partial R^2}\cdot\xi$$

$$\ddot{\eta} + 2\omega_o\dot{\xi} = 0$$

$$\ddot{z} = -\frac{\partial^2 U}{\partial z^2}\cdot z$$

Solution:

$$z = z_o\ \cos\omega_z\left(t - t_2\right)$$

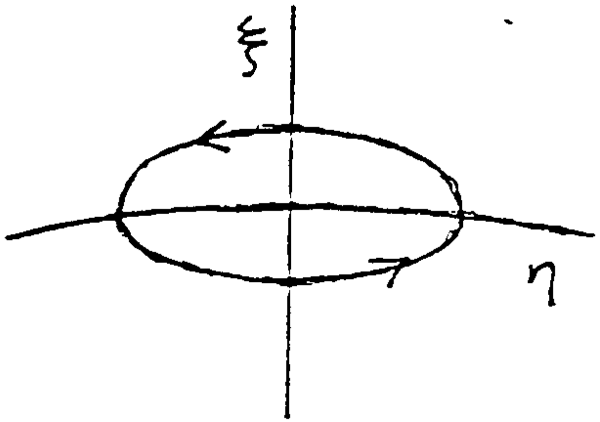
$$\xi = \frac{2\omega_o a}{\kappa_o^2} + c\ \cos\kappa_o\left(t - t_o\right)$$

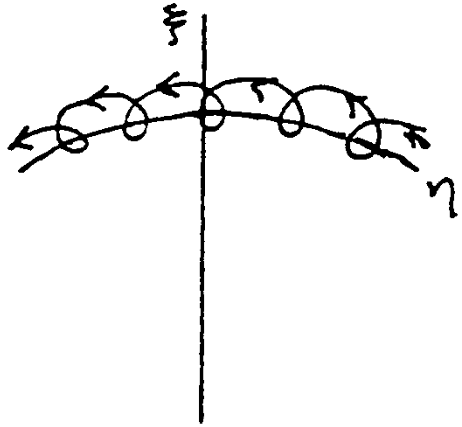
$$\eta = a\left(1 - \frac{4\omega_o^2}{\kappa_o^2}\right)\left(t - t_1\right) - \frac{2\omega_o c}{\kappa_o}\ \sin\kappa_o\left(t - t_o\right)$$

κ_o is the epicyclic frequency

$$\kappa_o^2 = \frac{\partial^2 U}{\partial R^2} + 3\omega_o^2 = -4\omega_o B$$

Epicyclic motion, ctd

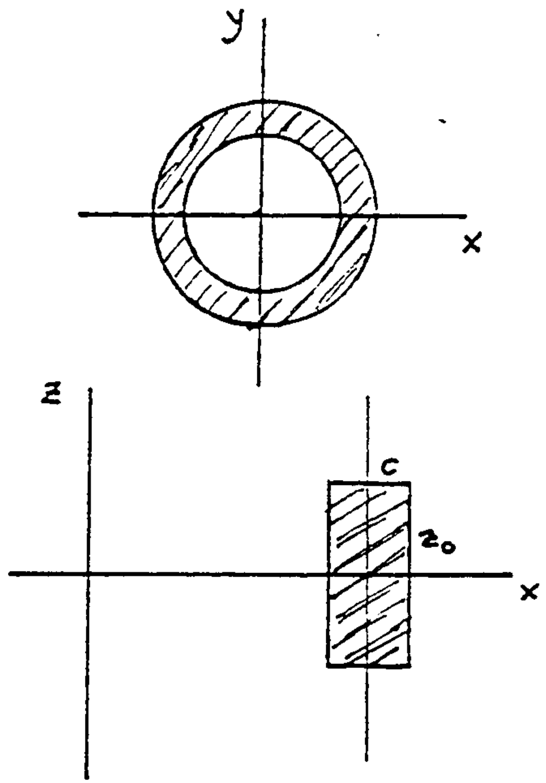




epicycle ***epicycle + drift***

The epicyclic motion combined with the asymmetric drift is an expression of a rosette orbit in a rotating frame

Orbits in the near-circular case



General orbit:

1) Circular rotation around the center
2) Small-scale epicyclic motion
3) Vertical oscillation

Three independent frequencies:

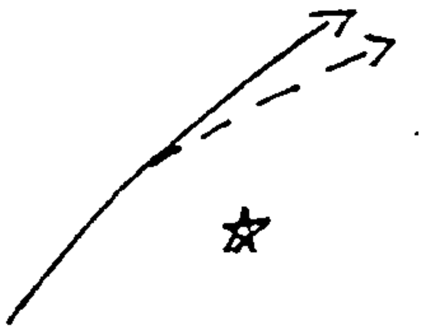
$$\omega_o^2 = \frac{1}{R}\frac{\partial U}{\partial R}$$

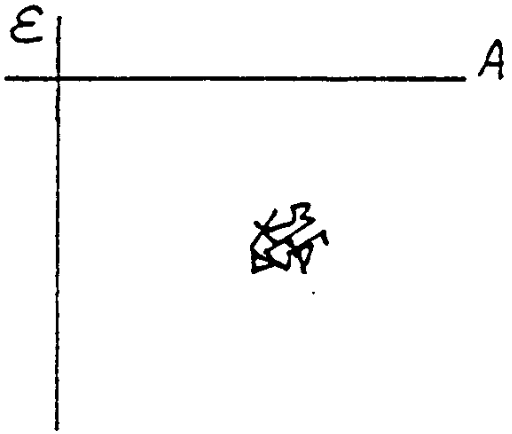
$$\kappa_o^2 = \frac{\partial^2 U}{\partial R^2} + \frac{3}{R}\frac{\partial U}{\partial R}$$

$$\omega_z^2 = \frac{\partial^2 U}{\partial z^2}$$

Two non-isolating integrals, the third integral is isolating

Effects of close encounters





- Consider a *test star* moving within an ensemble of *field stars*
- The motion of the test star is deflected at close encounters with field stars
- Thus the quantities defined by integrals are exchanged
- Each star performs a "*Brownian motion*" in the space spanned by the integrals

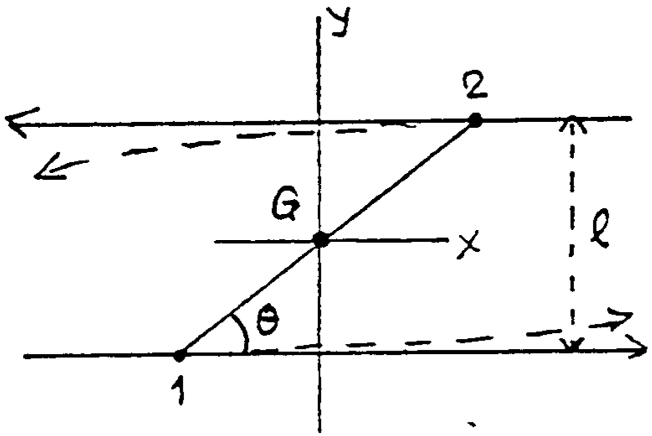
Collisional evolution

- Stars jump in and out of any phase space volume due to close encounters
- The net effect is an explicit *time dependence of the distribution function* and a *"collisional term" in the Boltzmann equation*
- Thus stars with given values of the integrals no longer have the same value of *f*, and *the stationary state is perturbed*
- But due to the rapid dynamical mixing a new stationary state will appear, and the system evolves by ***virialization*** and ***relaxation***

Quick! *Slow!*

Relaxation time

Approximate treatment



Velocity change of star 1:

$$\Delta \mathrm{V}_1 = \int_{-\infty}^{+\infty} \mathrm{a}_1 \, dt$$

Acceleration vector:

$$\mathrm{a}_1 = \frac{Gm \sin^2 \theta}{\ell^2} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$x_1 = -\frac{V}{2} t = \frac{\ell}{2} \cot\theta \qquad dt = \frac{\ell}{V} \sin^{-2}\theta \, d\theta$$

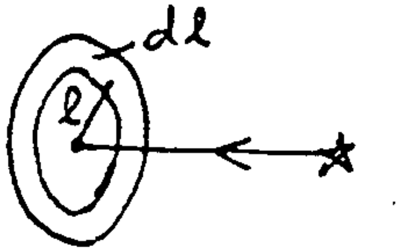
Variable transformation

Relaxation time, ctd

$$\Delta \mathrm{V}_1 = \frac{Gm}{\ell V}\int_0^{\pi}\left(\begin{array}{c}\cos\theta\\ \sin\theta\end{array}\right)d\theta \quad \Rightarrow \quad \begin{array}{l}\Delta u_1 = 0\\ \Delta v_1 = \dfrac{2Gm}{\ell V}\end{array}$$

deflection

Total effect during time Δt by random walk:



$$\left\langle (\Delta \mathbf{V}_1)^2 \right\rangle = \int (\Delta \mathbf{V}_1)^2\, dp$$

$$dp = \frac{\rho}{m}\cdot 2\pi\ell\, d\ell \cdot V\Delta t$$

This integral diverges logarithmically:

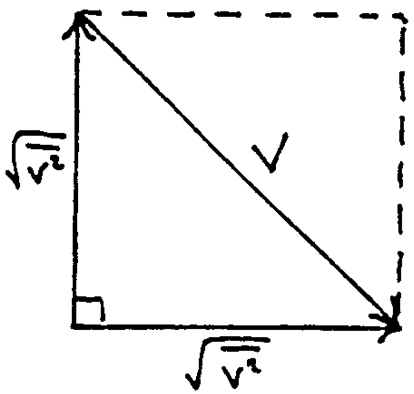
$$\left\langle \Delta v_1^2 \right\rangle = \int \frac{4G^2m^2}{\ell^2V^2}\cdot\frac{\rho}{m}\cdot 2\pi\ell d\ell\cdot V\Delta t \qquad \int_{\ell_{min}}^{\ell_{max}}\frac{d\ell}{\ell} = \ln\frac{\ell_{max}}{\ell_{min}}$$

replacement

Relaxation time, ctd

Limited system size: $\ell_{max} = R$

Maximum reasonable deflection: $\Delta v_1 = V \quad \Rightarrow \quad \ell_{min} = 2Gm/V^2$



Virial theorem: $\overline{v^2} = \frac{GM}{2R}$

$$V^2 = 2\overline{v^2} \quad \Rightarrow \quad \ell_{min} = \frac{2m}{M} R = \frac{2R}{N}$$

$$\ln \frac{\ell_{max}}{\ell_{min}} = \ln \frac{N}{2} \simeq \ln N$$

Relaxation time: $\Delta t = t_r \quad \Rightarrow \quad \left\langle \Delta v_1^2 \right\rangle \sim V^2$

Relaxation time estimates

$$t_r \sim \frac{V^3}{8\pi G^2 m\rho \ln N}$$

- For the *solar neighbourhood*, using $V \sim 30$ km/s and $\ln N \sim 10$ for the stellar population, we get $t_r \sim 10^{14}$ yr ***VERY LONG!***
- Relaxation effects like *increasing velocity dispersion and disk scale height for older populations* are observed
- These may be explained by the presence of massive objects like *star clusters and Giant Molecular Clouds*

Relaxation time estimates, ctd

- In comparison to the crossing time scale of a cluster, $t_c \sim R/V$:

$$\frac{t_r}{t_c} \sim \frac{V^4}{8\pi G^2 m \rho R \ln N}$$

$$\rho \sim \frac{M}{\frac{4}{3}\pi R^3} \simeq \frac{M}{4R^3}\,; \quad m \sim \frac{M}{N}$$

$$\frac{t_r}{t_c} \sim \frac{1}{2\pi} \cdot \frac{N}{\ln N}$$

- Thus, *the relaxation time is much larger than the crossing time, when N is very large*