STELLAR DYNAMICS

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These notes are intended as an introduction to fundamental concepts of stellar dynamics, and while observed stellar systems will often be taken as examples, the aim is not to describe those systems in detail but rather to focus on the dynamical processes.

1 Fundamental equations

1.1 The N-body problem

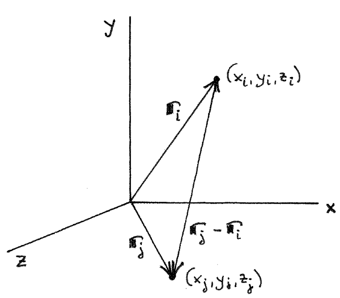
Consider a self-gravitating system of N objects that do not experience any external forces – *i.e.*, the system is *isolated*. The objects in question will always be stars, so we will use the term *stars* instead of 'objects'. Let us number the stars from 1 to N.

Consider a cartesian coordinate system x,y,z. We denote the coordinates of star i by x_i,y_i,z_i. These are functions of time, since the star is moving. The **velocity** has components

$$\begin{aligned} \frac{dx_i}{dt} &= \dot{x}_i = & u_i \\ \frac{dy_i}{dt} &= \dot{y}_i = & v_i \\ \frac{dz_i}{dt} &= \dot{z}_i = & w_i \end{aligned}$$

(the dots traditionally stand for time derivatives). For convenience we name them u_i,v_i,w_i. Finally, let the mass of star i be m_i. By $\mathbf{r}_i$ we denote the coordinate vector (x_i,y_i,z_i) that connects the origin to star i. We denote by r_{ij} the **distance** between stars i and j:

$$r_{ij} = |\mathbf{r}_j - \mathbf{r}_i| = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2} \tag{1}$$



The force of attraction between stars i and j has the absolute value Gm_im_j/r_{ij}^2 (G is the gravitational constant). Its direction is along the line i–j. Thus, in vector form, the force exerted by j on i is:

$$\frac{Gm_im_j}{r_{ij}^2} \cdot \frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|}$$

(the second factor is the unit vector from i toward j).

Hence the **total force** acting on star i due to all the others is:

$$m_i\ddot{\mathbf{r}}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{Gm_im_j(\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \qquad (i = 1, \ldots, N) \tag{2}$$

According to Newton's 2nd law of motion, this force equals the product of the acceleration and the mass (thus the above equation). This is a vector equation, which represents three scalar equations; the first is obtained by isolating the x components:

$$m_i \ddot{x}_i = \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G m_i m_j (x_j - x_i)}{r_{ij}^3} \qquad (i = 1, \ldots, N)$$

and similarly for y and z.

Equation (1) provides the foundation for the whole subject of stellar dynamics, as well as celestial mechanics. Such an equation can be written down for each star, *i.e.*, for all values of i, from 1 to N. We have a system of $3N$ simultaneous second-order differential equations and thus a dynamical system of order $6N$. If one knows the exact positions and velocities of all the stars at an initial moment, these equations in principle allow to compute the subsequent evolution of the stellar system for all times. However, such an exact solution is impossible to achieve by numerical integration owing to round-off errors and finite time steps of the integration. We hence wish to proceed as far as possible by analytic means.

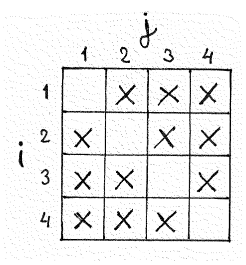
1.2 The ten integrals

One may obtain some important information about the system from the basic equations of motion without integrating them. Let us first seek **integrals** of the system, *i.e.*, entities that do not change with time.

1.2.1 Motion of the center of mass

Let us form the sum of the equations, from 1 to N:

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G m_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \qquad (i = 1, \ldots, N)$$



In the right-hand member we have all pairs (i, j) represented except those for which $i = j$. The sum of the two terms (i, j) and (j, i) is:

$$\frac{G m_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} + \frac{G m_j m_i (\mathbf{r}_i - \mathbf{r}_j)}{r_{ji}^3}$$

But $r_{ji} = r_{ij}$, and hence this sum equals zero. In this way the terms cancel out in pairs, and we have:

$$\sum_{i=1}^{N} m_i \ddot{\mathbf{r}}_i = \mathbf{0} \tag{3}$$

Physical Interpretation: Eq. (3) means that the sum of all forces within the system is zero, which is always true for an isolated system. If we integrate over time, we get:

$$\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = \mathbf{a},$$

where **a** is a constant vector, and then:

$$\sum_{i=1}^{N} m_i \mathbf{r}_i = \mathbf{a}t + \mathbf{b},$$

where $\mathbf{b}$ is yet another constant vector.
The **center of mass** of the system, $\mathcal{G}$, is defined by:

$$\mathbf{r}_{\mathcal{G}} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\mathbf{a}t + \mathbf{b}}{M} \tag{4}$$

where M is the total mass of the system, which is also a constant. This shows that the center of mass moves with constant velocity along a straight line, which is obvious, since no force is acting upon the system.

In mechanics we can always replace one frame of reference by another one in uniform, rectilinear motion with respect to the first. Hence, in all that follows, we shall take a coordinate system anchored at the center of mass, *i.e.*, which has its origin in $\mathcal{G}$ and moves with $\mathcal{G}$. In this system we thus have: $\mathbf{r}_{\mathcal{G}} \equiv \mathbf{0}$, *i.e.*: $\mathbf{a} = \mathbf{0}$ and $\mathbf{b} = \mathbf{0}$.

The choice of the center of mass frame simply means that we eliminate the *common motion* of the system, which does not interest us. We shall only investigate the *internal motions*.

1.2.2 Total angular momentum integral

Take the cross product of both members of the basic equation by $\mathbf{r}_i$, and form the sum over i:

$$\sum_{i=1}^{N} m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{G m_i m_j \mathbf{r}_i \times (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3} \qquad (i = 1, \ldots, N)$$

All the terms $\mathbf{r}_i \times \mathbf{r}_i$ in the right-hand member vanish. Let us again form pairs (i, j) and (j, i):

$$\frac{G m_i m_j \mathbf{r}_j \times \mathbf{r}_i}{r_{ij}^3} + \frac{G m_j m_i \mathbf{r}_i \times \mathbf{r}_j}{r_{ji}^3} = \mathbf{0},$$

since $\mathbf{r}_j \times \mathbf{r}_i = -\mathbf{r}_i \times \mathbf{r}_j$. Hence:

$$\sum_{i=1}^{N} m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = \mathbf{0} \tag{5}$$

Thus we have:

$$\frac{d}{dt}\left(\sum_{i=1}^{N} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i\right) = \sum_{i=1}^{N} m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i = \mathbf{0}$$

which yields:

$$\sum_{i=1}^{N} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \mathbf{c} \tag{6}$$

where, again, $\mathbf{c}$ is a constant vector.

Physical Interpretation: This quantity is the **total angular momentum** of the system, and it is conserved for an isolated system, on which no external torque is acting.

It cannot be eliminated by a change of reference system, like we did with $\mathbf{a}$ and $\mathbf{b}$. In other words, $\mathbf{c}$ is an invariant vector that characterizes the system. In particular, it defines a **characteristic direction**. If the system has an axis of symmetry, this axis should coincide with $\mathbf{c}$. We shall also see below that the *absolute value* of $\mathbf{c}$, which measures the amount of common rotation in the system, is linked to the **flattening** of the system.

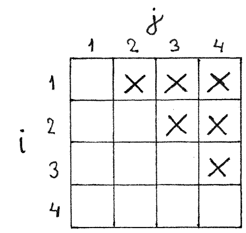
1.2.3 Total energy integral

Let us define a pairwise *potential energy* of stars i and j as the quantity:

$$\Omega_{ij} = -\frac{Gm_i m_j}{r_{ij}}$$

and the *total potential energy* as the sum of all pairwise energies, *i.e.*:

$$\Omega = -\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{Gm_i m_j}{r_{ij}} \tag{7}$$



Each pair should appear once and only once, which explains the definition of the sums. It can be shown that Ω is the work performed by the system, if all the stars are removed infinitely far from each other. Thus the name **potential energy**, *i.e.*, work that potentially may be performed. In reality Ω is negative, *i.e.*, work must be provided to the system in order to push the stars away from each other. This is obvious, since they are attracting each other.

Ω is a function of all the coordinates: $\Omega(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_N, y_N, z_N)$ – these appear in the expressions for r_{ij}. Let us differentiate with respect to one of them and form $\partial\Omega/\partial x_i$. Only the pairs formed by star i and another star yield a non-vanishing derivative. Thus:

$$\frac{\partial \Omega}{\partial x_i} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{\partial}{\partial x_i}\left(\frac{Gm_i m_j}{r_{ij}}\right) = \sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{Gm_i m_j}{r_{ij}^2} \frac{\partial r_{ij}}{\partial x_i}$$

We have: $r_{ij}^2 = (x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2$, and hence:

$$\frac{\partial r_{ij}}{\partial x_i} = -\frac{x_j - x_i}{r_{ij}}$$

and:

$$\frac{\partial \Omega}{\partial x_i} = -\sum_{\substack{j=1 \\ j\neq i}}^{N} \frac{Gm_i m_j (x_j - x_i)}{r_{ij}^3}$$

But we saw before that this expression is exactly: $-m_i\ddot{x}_i$. Thus:

$$m_i \ddot{x}_i = -\frac{\partial \Omega}{\partial x_i}$$

and similarly:

$$m_i \ddot{y}_i = -\frac{\partial \Omega}{\partial y_i}$$
$$m_i \ddot{z}_i = -\frac{\partial \Omega}{\partial z_i}$$

which we merge into:

$$m_i \ddot{\mathbf{r}}_i = -\boldsymbol{\nabla}_i \Omega \tag{8}$$

(a notation that is not strictly correct but convenient). So we have yet another quite general result: **The force equals minus the gradient of the potential energy**.

Let us take the scalar product of both members with $\dot{\mathbf{r}}_i$ and form the sum over i:

$$\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i + \sum_{i=1}^{N} \dot{\mathbf{r}}_i \cdot \boldsymbol{\nabla}_i \Omega = 0 \tag{9}$$

or:

$$\frac{d}{dt} \sum_{i=1}^{N} \frac{m_i \dot{\mathbf{r}}_i^2}{2} + \frac{d\Omega}{dt} = 0 \tag{10}$$

since the second term of Eq. (9) can be written:

$$\frac{\partial \Omega}{\partial x_1} \dot{x}_1 + \ldots + \frac{\partial \Omega}{\partial z_N} \dot{z}_N = \frac{\partial \Omega}{\partial x_1} \frac{dx_1}{dt} + \ldots + \frac{\partial \Omega}{\partial z_N} \frac{dz_N}{dt} = \frac{d\Omega}{dt}$$

The sum in the first term of Eq. (10) is called the total **kinetic energy** (T) of the system, *i.e.*, the sum of the individual kinetic energies of the stars. We integrate over time and obtain:

$$\boxed{T + \Omega = E} \tag{11}$$

where E is a constant called the **total energy**. This is simply the theorem of energy conservation.

1.2.4 Concluding remarks

We have found a number of constant quantities – the vectors **a**, **b** and **c** plus the scalar E. Since each vector has three components, this yields ten constants, *i.e.*, ten **integrals** of the system of differential equations. In principle, each integral allows to substitute one variable in terms of the others, *i.e.*, to reduce the order of the system by 1. Thus the order of the system can be reduced from $6N$ to $6N - 10$.

In the simplest of all cases – the two-body problem ($N = 2$) – it is actually possible to solve the problem completely. But as soon as $N \geq 3$, such a solution would mean finding ≥ 18 integrals, and this has not been possible so far. And for many-body systems like stellar clusters with $N \sim 10^3 - 10^6$ or galaxies with $N \sim 10^{11}$ it does not make sense at all to hope for analytic solutions. This is one fundamental reason why we have to discuss the behaviour of such systems in terms of **statistical mechanics**.

Finally, note that the integrals derived above refer to the whole system and not to individual stars. In the most general case, the energy or angular momentum of an individual star will not be conserved. However, we shall see that there are cases, where the symmetry properties of the systems allow the existence of such integrals, and that this is very useful for constructing models of those systems.

1.3 The virial theorem

1.3.1 Lagrange's identity

Yet another very important property – although not an integral – can be derived from the equations of motion. Let us consider the quantity:

$$J = \sum_{i=1}^{N} m_i \mathbf{r}_i^2, \tag{12}$$

which is called the **moment of inertia** about the origin. Note that in mechanics in general, the moment of inertia is taken with respect to an *axis* so that $\mathbf{r}_i$ is replaced by the distance from this axis. The quantity J that we now consider is thus a little different.

Let us take the first and second time derivatives of J as follows:

$$\dot{J} = 2\sum_{i=1}^{N} m_i \mathbf{r}_i \cdot \dot{\mathbf{r}}_i$$

$$\ddot{J} = 2\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i^2 + 2\sum_{i=1}^{N} m_i \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i$$

which may be written:

$$\ddot{J} = 4T + 2\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} \frac{Gm_i m_j \mathbf{r}_i \cdot (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}$$

In the second part of the right-hand member we again bring together the terms (i,j) and (j,i):

$$\mathbf{r}_i \cdot (\mathbf{r}_j - \mathbf{r}_i) + \mathbf{r}_j \cdot (\mathbf{r}_i - \mathbf{r}_j) = -(\mathbf{r}_j - \mathbf{r}_i)^2 = -r_{ij}^2$$

and from this we conclude that:

$$\ddot{J} = 4T - 2\sum_{i=1}^{N}\sum_{j=i+1}^{N} \frac{Gm_i m_j}{r_{ij}},$$

i.e.,

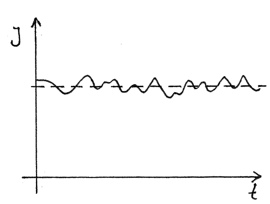
$$\boxed{\ddot{J} = 4T + 2\Omega} \tag{13}$$

This relation is known as **Lagrange's identity**.

1.3.2 Virial equilibrium

We have just obtained a result of great practical importance. We shall see below that stellar systems in general are in **steady state**. Thus, despite the fact that the stars keep moving within the system, the bulk properties of the system do not change. For instance, the extent – or size – of the system remains constant. But the quantity J is connected to the extent of the system: if this would be increasing, then all the r_i and therefore also J would increase. Thus, in vague terms, the steady state means that:

$$J = constant$$



Strictly speaking, due to the motions of individual stars, J cannot be exactly constant but exhibits **statistical fluctuations** around an average. Hence, let us instead write: $\overline{J} = const.$, where $\overline{J}$ stands for a time average of J smoothing out the short-term fluctuations. Since, as we shall see later, the steady states of stellar systems do not last forever, but the systems evolve over long time scales, we may consider averaging over an interval that is long enough to eliminate the fluctuations but still short in comparison to the evolutionary time scale of the system.

Taking time derivatives, we realize that the averages of $\dot{J}$ and $\ddot{J}$ both vanish. Thus, by averaging Lagrange's identity, we find:

$$4\overline{T} + 2\overline{\Omega} = 0$$

From the energy integral we know that

$$T + \Omega = E,$$

which holds strictly at each moment. Forming the average, we find:

$$\overline{T} + \overline{\Omega} = E$$

By solving the two above equations for $\overline{T}$ and $\overline{\Omega}$, we obtain:

$$\overline{T} = -E \qquad ; \qquad \overline{\Omega} = 2E \tag{14}$$

which shows that the time averages of T and Ω are also constant.

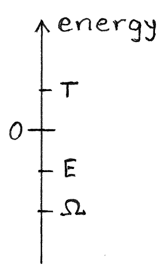
The importance of fluctuations depends on the number of stars. It can be shown that in a system of N stars, the relative amplitude of the statistical fluctuations is $N^{-1/2}$. In relatively small-number systems like open star clusters, the fluctuations can be sizeable, but in globular clusters or galaxies they are negligible. In the latter cases we can thus write T and Ω instead of their averages, *i.e.*:

$$\boxed{T = -E \qquad ; \qquad \Omega = 2E} \tag{15}$$

and:

$$\boxed{2T + \Omega = 0} \tag{16}$$

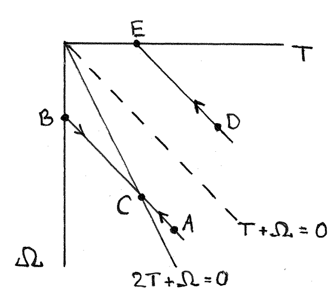
which is known as the **virial theorem**.



The relative positions of the energies should be remembered, The three intervals are equal. This repartition is generally valid in all stationary systems. In physical terms, the virial theorem suggests that there is a certain equilibrium between the kinetic and potential energy. This is easily understood. The kinetic energy tends to remove the stars from each other because of their motions, and by contrast the potential energy is linked to the forces of attraction and thus tends to bring the stars closer together. We realize that if the system is to remain in a steady state, there has to be an equilibrium between these two tendencies. This is expressed mathematically by the virial theorem.

We may ask if this equilibrium is **stable**? More generally, suppose that a system does *not* obey the virial theorem at the initial moment. How will it evolve?

Consider a diagram with T as abscissa and Ω as ordinate. We always have: $T \geq 0$ and $\Omega \leq 0$. At each moment the system is represented by a point in this diagram, and this point moves with time. We have: $T + \Omega = E = const.$; thus the point can only move along a straight line with slope -1. At the same time, a system obeying the virial theorem is situated on the line $2T + \Omega = 0$. Let us take as example a system starting at point A.



We have: $2T + \Omega > 0$, and thus $\ddot{J} > 0$, so that $J(t)$ has a positive curvature. Therefore, at least after some time, J will be increasing. This means that the extent of the system increases. We easily realize this physically, because the stars have too much kinetic energy, so they move too quickly and recede from each other. As they do this, they are decelerated by the forces of attraction. Thus the velocities – and hence T – will decrease. At the same time $|\Omega|$ decreases, since the distances increase. This sets the direction of motion in the (T, Ω) plane. The system moves toward the equilibrium point C.

Let us instead assume that the system starts on the other side with $2T + \Omega < 0$, for example at point B. In this case the initial value of T is zero, *i.e.*, all the stars are at rest. Consequently they will "fall" toward each other, so that T increases and $|\Omega|$ increases. The motion is again directed toward C. Thus any system tends to tune its extent so that the virial theorem is obeyed. The process of doing so for a system that is initially not in virial equilibrium is sometimes called **virialization**. In reality, the representative point will describe a damped oscillation around C. However, the oscillations of stellar systems are damped very quickly, so the observed star clusters, for instance, are generally in steady state. Their ages are far larger than the virialization time.

A system can reach equilibrium only if it is situated below the dashed line. If, *e.g.*, it starts from point D, it will not reach the equilibrium line but ends up at E, where $\Omega = 0$ and the stars have moved infinitely far apart. Thus the system is dissolved and ceases to exist. We see that **a necessary condition for stability of a stellar system is that its total energy is negative**. Some young "stellar associations" in the Galaxy in fact have positive energies. They are thus temporary, new-born clusterings of young stars that have not had the time to dissolve but are in the process of dissolving.

1.3.3 Using the virial theorem

The practical use of the virial theorem comes from the fact that it yields a relation between the essential physical parameters of stellar systems. T depends on the velocities and the total mass, and Ω depends on the extent and the mass. Thus we have got a relation between the mass, extent and velocities, and if we observe two of these properties, we can derive the third using the virial theorem.

To formulate this relation, let us first write:

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 = \frac{1}{2}N\langle mv^2\rangle$$

where $v_i = |\dot{\mathbf{r}}_i|$ is the absolute value of the velocity of a star, and the average "$\langle\ \rangle$" is taken over the whole set of stars.

Now, we would like to write $\langle mv^2\rangle \simeq \langle m\rangle\langle v^2\rangle$. We must realize that this may not be a very good approximation in case there is a correlation between the masses and velocities of the stars (and in reality these are indeed correlated!), but let us use it all the same. At the same time we have:

$$\Omega = -\sum_{i=1}^{N-1}\sum_{j=1+1}^{N}\frac{Gm_i m_j}{r_{ij}} \simeq -\frac{N(N-1)}{2}G\left\langle\frac{m_i m_j}{r_{ij}}\right\rangle$$

and – analogous to the above assumption – we put: $\langle m_i m_j/r_{ij}\rangle \simeq \langle m\rangle^2/\langle r\rangle$, where $\langle r\rangle$ is the mean distance between two stars of the system, randomly selected. This is of the order of the mean radius of the system. One can replace $N-1$ by N, since this is a large number, and the virial theorem hence yields:

$$N\langle m\rangle\langle v^2\rangle - \frac{N^2}{2}G\frac{\langle m\rangle^2}{\langle r\rangle} \simeq 0$$

We have: $N\langle m\rangle = M$, the total mass, which yields:

$$\langle v^2\rangle \simeq \frac{GM}{2\langle r\rangle} \tag{17}$$

This is the "practical" form of the virial theorem. One should keep in mind that it only provides an approximate relation.

Examples. Consider an open star cluster with 500 stars of $\langle m \rangle = 1\ M_\odot$. The radius of the cluster is $\langle r \rangle = 1$ pc. If we measure masses in solar masses, distances in pc, and times in years, we have: $G = 4.5 \cdot 10^{-15}$, so we get from Eq. (17):

$$\langle v^2 \rangle = 4.5 \cdot 10^{-15} \cdot 500\ (\mathrm{pc/yr})^2 = 2.25 \cdot 10^{-12}\ (\mathrm{pc/yr})^2,$$

and since 1 pc/Myr is very close to 1 km/s (very useful to remember!), the square root of $\langle v^2 \rangle$ turns out to be 1.5 km/s. This is a measure of the **velocity dispersion** of the cluster, or the typical value of the relative velocity of any pair of stars within the cluster. Note how small this is compared to the typical velocities of the stars in the solar neighbourhood relative to the Sun, which are ~ 30 km/s. But those velocities are a different matter, since the stars of the solar neighbourhood do not form a self-gravitating system but are part of the whole Galaxy. We will come to Galactic dynamics later on.

Another example concerns globular clusters, whose radii are more like 10 pc, and which are made up of $\sim 10^6$ stars with an average mass of $0.5\ M_\odot$. In this case we get a velocity dispersion of about 4.7 km/s.

Finally, consider galaxy clusters, for which we easily observe the radii and velocity dispersions but have to derive the masses from the virial theorem. As typical estimates for radii and velocity dispersions we can take 0.5 Mpc and 1000 km/s, which yields $M \sim 2 \cdot 10^{14}\ M_\odot$. The classical problem here is that adding up the estimated masses of the member galaxies, we get a much smaller value. This has been referred to as the **missing mass** problem of galaxy clusters, and it provides one of the best proofs of the existence of **dark matter** in the Universe.

1.4 Statistical description of stellar systems

1.4.1 The distribution function

Consider the state of the system at a given moment t_0. A complete description of this state involves knowing for each star i: the mass m_i, the position (x_i, y_i, z_i), and the velocity (u_i, v_i, w_i). In principle, from this information one may compute the evolution of the system for all times.

Let us now consider a 7-dimensional space called the **phase space**. Each star is represented in this space by a point, whose seven coordinates are:

$$x_i, y_i, z_i, u_i, v_i, w_i, m_i$$

Thus the whole system is represented by a set of N points in phase space. Conversely, this set completely describes the state of the system.

In general the number of points is very large. Thus it is not possible in practice to specify the exact position of each representative point. But exactly because N is large, we may replace the exact description of the system by a **statistical description**. Instead of specifying the exact positions of the N points, we specify only the density of such points in phase space. This density is defined as follows.

We divide the phase space into small cells. Each cell has to be small compared to the extent of the whole set of points but yet large enough to contain a considerable number of points. Let $\Delta\tau$ be the volume of a cell (in seven dimensions) and Δn the number of points within the cell. The density associated with this cell is then given by:

$$\Psi = \frac{\Delta n}{\Delta \tau} \tag{18}$$

We thus get a value of Ψ for each cell, and by interpolation and smoothing we may obtain a *continuous function*, defined everywhere in phase space. Note that this function Ψ is not rigorously defined but only approximately. Its approximate definition becomes better, the larger the number N of points.

Ψ is a function of the seven phase space coordinates but also depends on time, because the representative points are moving as the stars follow their orbits:

$$\Psi(x, y, z, u, v, w, m, t)$$

This is called the **distribution function**. According to its definition, it obeys the condition:

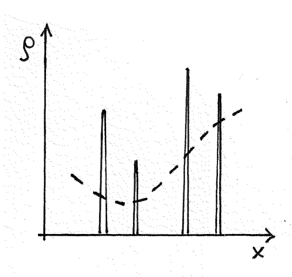
$$\int\int\int\int\int\int\int \Psi\, dxdydzdudvdwdm = \int \Psi\, d\tau = N$$

1.4.2 Smoothed density and potential

Let us now compute the usual **density** of the stellar system in 3-dimensional physical space (x, y, z). It is derived from Ψ simply by summing, or integrating, over all values of velocity and mass, *i.e.*:

$$\rho(x, y, z, t) = \int_0^{\infty} m\, dm \int\int\int_{-\infty}^{+\infty} \Psi\, dudvdw \tag{19}$$

Each star contributes with its mass m such that ρ is the density in the classical sense: the mass per unit volume, while Ψ instead was a number of points per unit volume in phase space. Note that ρ depends on time. We see that ρ can be derived from Ψ, but the converse is not true. Knowledge of the usual density ρ is not enough to define the system.



The function ρ is called **smoothed density**. It is a continuous function, while the actual density is markedly discontinuous, being zero everywhere except in the stars, where it is extremely high. The sketch to the left indicates a one-dimensional system for simplicity. The smoothed density is obtained by eliminating the local roughness as part of the smoothing done for the distribution function. We may imagine that the material in each star has been spread out into a volume around the star so that the density curve has been smoothed.

The **gravitational potential** that is set up at any point by the whole stellar system is given by the classical formula:

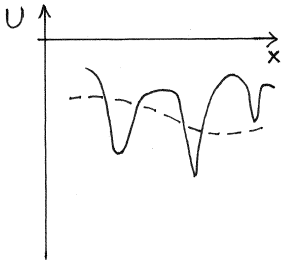
$$\boxed{U(x_1, y_1, z_1, t) = -G \int\int\int_{-\infty}^{+\infty} \frac{\rho(x_2, y_2, z_2, t)\, dx_2 dy_2 dz_2}{r_{12}}} \tag{20}$$

where r_{12} is the distance between points 1 and 2.

This relation allows to compute U from ρ. It may be inverted, and one then obtains:

$$\boxed{4\pi G\rho = \boldsymbol{\nabla}^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}} \tag{21}$$

which is called the **Poisson equation**. Generally speaking, it is easier to use than Eq. (20), even for computing U from ρ, since it contains no singularity. But, of course, when integrating Eq. (21), we have to take care about the boundary conditions: $U = 0$ at infinity.



U is called the **smoothed potential**, since it is obtained from the smoothed density. The real potential (solid curve in the sketch to the left) exhibits deep wells in the vicinity of all stars. If a star enters into such a potential well, its motion will be very different from the one due to the smoothed potential, and this phenomenon will be analysed below.

1.4.3 The Liouville equation

As a first approximation, supposing the number of stars is large, we may disregard the local irregularities of the real potential and assume that it is well represented by the smoothed potential U. The motion of a star is then given by:

$$\ddot{x} = -\frac{\partial U}{\partial x} \quad ; \quad \ddot{y} = -\frac{\partial U}{\partial y} \quad ; \quad \ddot{z} = -\frac{\partial U}{\partial z} \tag{22}$$

Each one of these second-order differential equations can be broken down into two first-order equations, and we thus get:

$$\left.\begin{array}{rcl} \dot{x} = & u & = f_x \\ \dot{y} = & v & = f_y \\ \dot{z} = & w & = f_z \\ \dot{u} = & -\partial U/\partial x & = f_u \\ \dot{v} = & -\partial U/\partial y & = f_v \\ \dot{w} = & -\partial U/\partial z & = f_w \\ \dot{m} = & 0 & = f_m \end{array}\right\} \mathbf{f} \tag{23}$$

The last equation expresses the fact that the mass of a star remains constant in the course of time. These seven equations describe the motion of a representative point in phase space. The **velocity** of the point is a vector $\mathbf{f}$ with components $(f_x, \ldots, f_m)$. Let us call **trajectory** the curve described by a representative point in phase space in order to distinguish it from the **orbit**, which is a curve in physical space followed by the actual star.

If the state of the system is known, the potential U is also known, since it can be computed using the above equations. Hence $\mathbf{f}$ is known at each point in phase space. In other words, the motion of the representative points in phase space resembles the motion of a *fluid* – at each point the velocity is a uniquely defined vector! Note that this is not true at all for the 3-dimensional physical space, since at any given point the stars may have arbitrary velocities.

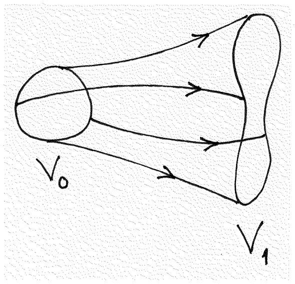
The vector field $\mathbf{f}$ has the important property:

$$\boxed{\text{div } \mathbf{f} = 0} \tag{24}$$

which is easily proved:

$$\text{div } \mathbf{f} = \boldsymbol{\nabla} \cdot \mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} + \frac{\partial f_u}{\partial u} + \frac{\partial f_v}{\partial v} + \frac{\partial f_w}{\partial w} + \frac{\partial f_m}{\partial m}$$

where $f_x, \ldots$ are the components of $\mathbf{f}$. But $f_x = u$, and thus: $\partial f_x/\partial x = 0$ (*Note:* u and x are independent coordinates in phase space). Likewise for the two following terms. Moreover, $f_u = -\partial U/\partial x$ depends only on x, y, z, and thus: $\partial f_u/\partial u = 0$. Likewise for the two following terms. Finally, $f_m \equiv 0$. Hence each of the terms individually equals zero.



Geometric interpretation: The ensemble of representative points behaves like an **incompressible fluid**. Consider a volume V_0 (with seven dimensions) at time t_0. At some later time t_1 the points that were inside V_0 have moved and now occupy a volume V_1. But it is a fact of geometry that, if the flow of points is divergence free, *i.e.*, $div\ \mathbf{f} = 0$, then $V_1 = V_0$. The volume may have changed shape, but its measure is unchanged. Thus:

Phase space volumes are conserved during the motion of the system.

Let us also consider the stars, whose points were inside V_0 at time t_0. We denote their number by n. At time t_1 these stars, *and only these*, yield points within V_1. This follows from the fact that phase space trajectories cannot cross, when the motion is governed by a smooth potential. The value of Ψ in V_0 at time t_0 is n/V_0, and the value of Ψ in V_1 at time t_1 is n/V_1. These two values are identical, and thus we have the following fundamental result:

The value of $\boldsymbol{\Psi}$ is constant along phase space trajectories.

The last statement is called **the Liouville theorem**.

We may also formulate the above arguments by means of equations. The principle of mass conservation can be written for any fluid:

$$\frac{\partial \Psi}{\partial t} + \operatorname{div}(\mathbf{f}\Psi) = 0$$

Developing the second term, we get:

$$\frac{\partial \Psi}{\partial t} + \mathbf{f} \cdot \boldsymbol{\nabla}\Psi + \Psi \operatorname{div} \mathbf{f} = 0$$

The third term vanishes according to the above result. If we develop the scalar product, we get:

$$\frac{\partial \Psi}{\partial t} + f_x \frac{\partial \Psi}{\partial x} + \cdots + f_m \frac{\partial \Psi}{\partial m} = 0, \quad i.e.:$$

$$\boxed{\frac{\partial \Psi}{\partial t} + u\frac{\partial \Psi}{\partial x} + v\frac{\partial \Psi}{\partial y} + w\frac{\partial \Psi}{\partial z} - \frac{\partial U}{\partial x}\frac{\partial \Psi}{\partial u} - \frac{\partial U}{\partial y}\frac{\partial \Psi}{\partial v} - \frac{\partial U}{\partial z}\frac{\partial \Psi}{\partial w} = 0} \qquad (25)$$

which is called the **Liouville equation** or the **collision-free Boltzmann equation**. The meaning of the term 'collision-free' will be explained in the next subsection. This equation expresses mathematically the fact that Ψ is constant when following the motion. Specifically, the time variation of Ψ, when following the motion, is given by:

$$\frac{d}{dt}\Psi(x, y, z, u, v, w, m, t) = \frac{\partial \Psi}{\partial x}\frac{dx}{dt} + \frac{\partial \Psi}{\partial y}\frac{dy}{dt} + \cdots + \frac{\partial \Psi}{\partial m}\frac{dm}{dt} + \frac{\partial \Psi}{\partial t}$$

i.e., the above expression.

The Liouville equation along with Eqs. (20) and (21) allow us (still, in principle) to compute the time evolution of the distribution function Ψ starting from a given initial condition. In specific terms, if we know Ψ at $t = t_0$, we can derive ρ and then U, thus yielding $\partial\Psi/\partial t$, from which we can compute Ψ at the following moment $t_0 + dt$, etc.

In conclusion: When we replace the exact description of the system (as given by all the coordinates) by the statistical description in terms of the distribution function, the set of Newtonian equations of motion is replaced by the three above equations: the *density equation*, the *Poisson equation*, and the *Liouville equation*.

1.4.4 The effects of close encounters

In practice, our definition of the distribution function is not entirely satisfactory. Let us divide the phase space into small cells and count the stars in each cell in order to estimate Ψ. In order for Ψ to be well determined, we need a sufficient number of cells – say, at least 10 intervals along each coordinate axis. So we may use 10 intervals of x, 10 intervals of y, ..., up to m. With seven dimensions we thus get 10^7 cells. Moreover, we do not want the values of Ψ to be too much affected by statistical noise. In order to get Ψ to a statistical accuracy of $\sim 10\%$, we need ~ 100 points per cell.

In summary, we need at least $\sim 10^9$ stars, and this is just to get a crude picture of Ψ. In reality stellar systems often contain much fewer stars. Actually, the picture used for Ψ comes from kinetic gas theory. In that case the number of particles is $\sim 10^{24}$, and thus Ψ is well determined. But within stellar dynamics we may ask if the definition is indeed meaningful. It is at least not rigorous!

A way out might be to leave the concept of phase space *density* in favour of the **probability** to find a star at any particular place. This means to use, for each star i, a function F_i defined so that:

$$F_i(x, y, z, u, v, w, m, t)\, dxdy \cdots dm$$

is the probability for the star to find itself in the volume element $dxdy \cdots dm$ around the point $(x, y, \ldots, m)$. We may write $F_i(\tau, t)\, d\tau$, where τ represents the set of seven coordinates, and $d\tau$ is the 7-dimensional volume element. The integral of each such *probability function* is of course unity, and there are N functions, one for each star.

For a theoretician such probability functions have the advantage of being rigorously defined independent of the number of stars in the system. But of course, in practice the probabilities have to be measured using a finite number of stars, so the practical problem cannot be escaped. Let us instead use the probability functions to illustrate the effects of direct interactions between individual stars, arising from **close encounters**.

The N functions F_i are not enough to describe the system completely. For instance, the probability that star i is in a volume $d\tau$ around τ and star j is in a volume $d\tau'$ around τ' would be equal to the product of $F_i(\tau, t)d\tau$ and $F_j(\tau', t)d\tau'$ if and only if the locations of the two stars were independent of each other. Thus, if we introduce the double probability function F_{ij}, we can write:

$$F_{ij} = F_i\, F_j + \epsilon_{ij} \tag{26}$$

Here ϵ_{ij} is the *correlation term.* It would be zero, if there was no correlation between stars i and j. One can show that the size of the correlation term is generally dictated by: $\epsilon_{ij}/F_iF_j \sim 1/N$, *i.e.*, if the number of stars is large, the correlations are generally weak. Thus, in this case, stars move *nearly independent* of each other.

This has to do with the smoothness of the potential function. The stars feel each others' presence, when they pass close and thus enter into each others' potential wells, and their motions are deflected. In sparse systems the potential is not very smooth, and the actual positions of the individual stars are important for determining the acceleration that any star experiences. But when the systems contain enormous numbers of stars, the smoothed potential gives quite a good description.

In fact, stellar dynamics can be divided into two parts that are very different from each other. The first part is relevant when considering the systems over short enough time scales that close encounters can be neglected, the correlation term is zero, and stars move independent of each other. In this case the smoothed density and potential functions are relevant, and the collision-free Boltzmann equation can be used. Alternatively, one can write down corresponding equations for the probability functions, and higher-order multiple functions are easily obtained from the first-order ones by utilizing Eq. (26) and neglecting the ϵ terms.

The second part is much more complicated. This is where the evolution of the system is governed by effects of close encounters – so-called **relaxation** effects. One way to describe this is to say that the correlation terms can no longer be neglected. They may be small, but they have a cumulative effect over a long time, so that the probability functions evolve in a distinctly different way. Another way to describe it is to realize that there are singularities in the potential function corresponding to the potential wells of individual stars, and when a star passes through such a well, its motion is affected by a deflection of its velocity vector. This happens very quickly, so the star effectively "jumps" from one phase space trajectory to another one. The corresponding thing of course happens to the other star, so both stars are jumping.

Thus the Liouville equation is no longer valid, because stars are able to jump into and out of any phase space volume because of close encounters – or **collisions**, as they are sometimes erroneously referred to. If we want to take those effects into account while using the statistical theory, we should replace 0 in the right-hand member of Eq. (25) by

$$\left(\frac{\partial\Psi}{\partial t}\right)_{coll},$$

and the expression for this extra term is unfortunately complicated. The critical time scale, over which relaxation effects become important for a stellar system, is called the **relaxation time**.

In the following we shall deal, in turn, with the dynamics of collision-free and collisional systems.

2 Collision-free systems

2.1 Dynamical mixing

The first question we ask is about time scales. In particular, what is the time typically required for a stellar system to change its distribution function considerably? More specifically, if a system starts from a configuration that is not in virial equilibrium, how long does it take for the distribution function to approach the steady state?

Essentially, it is a question of individual stars traversing the system and thereby exchanging their potential and kinetic energies. Therefore, the fundamental time scale that we look for must be the one of orbital motion within the system. This can be expressed by using the mean radius $\langle r \rangle$ (cf. Sect. 1.3.3) and the mean velocity, which we can take to be: $\langle v \rangle \simeq \sqrt{\langle v^2 \rangle}$. We thus get:

$$t_c = \frac{\langle r \rangle}{\langle v \rangle} \tag{27}$$

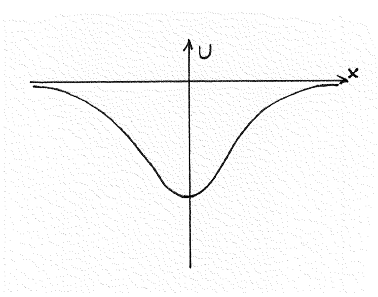
and t_c is usually called the **crossing time**, *i.e.*, the time it takes an average star to cross the system.

We use the practical form of the virial theorem in Eq. (17), and hence obtain the crossing time as:

$$t_c = \sqrt{\frac{2\langle r \rangle^3}{GM}} \tag{28}$$

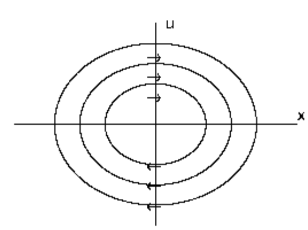
as a function of the mass and size of the system. Using the previous estimates for open and globular clusters, we derive $t_c \sim 1$ Myr in both cases. This is in general much less than the age of these clusters, so we conclude that they should have had more than time enough to evolve into equilibrium configurations. Indeed, the regular shapes observed in particular for globulars verify this expectation.

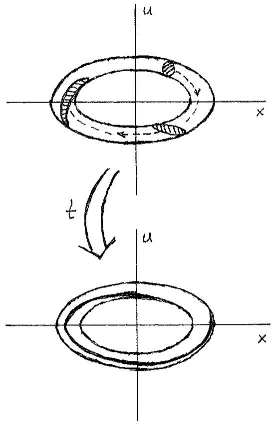
Now, let us illustrate the phenomenon of **phase space mixing** that is intimately connected with the settling into a steady state. For such an illustration we take a very simple example, *i.e.*, a system whose motions occur in only one dimension (of course, not realistic at all!) If we do not care about the masses of the objects, the phase space is then 2-dimensional, and we denote the coordinates by x and u.



Let us further assume that the potential U is given and has the shape indicated to the left. This corresponds to an attractive force toward the origin, and the system of our example is similar to that of a pendulum. The objects of this system will therefore oscillate around the origin with different amplitudes, periods and phases. The orbit in this case is a rectilinear segment along the x axis.

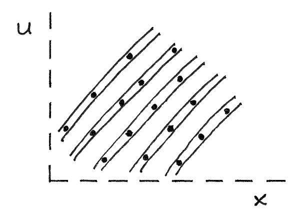
The trajectory, on the other hand, is an oval curve in the (x, u) plane. If we start with $x > 0$, $u = 0$, the object starts moving to the left with $u < 0$ while x decreases, and so on. The trajectory becomes a closed curve because of energy conservation. Starting from another point, we get a different trajectory with the same general appearance – an oval pattern described in the same way. The periods of different trajectories will in general be different.





Let us now consider a parcel of objects ("stars") that initially have approximately the same position and velocity – they cover a small area in phase space. In the course of time this parcel will move in the direction of the arrows, and thus the state is clearly non-stationary. As it moves, it also gets deformed due to the *differences in period* between different stars. Note that, in spite of this deformation, the area of the parcel remains the same according to the Liouville theorem. After one full revolution, it may have become significantly extended, and after another few revolutions the extension becomes extreme. Eventually the parcel will look like a narrow, winding spiral.

As this spiral continues to wind, the different rounds get squeezed closer together, and since the number of stars is finite, after some time it becomes in practice impossible to separate the rounds from each other. For all practical purposes, the distribution of stars looks uniform within the band defined by the initial parcel. We have a distribution function that is constant along the trajectories and does not change with time. This is the stationary state that corresponds to "*virial equilibrium*".



We can learn from this example that the establishment of a stationary state is accomplished by phase space mixing, which may be loosely called "*virialization*". After some number of crossing times, we may consider that any stellar system will be essentially in a stationary state, which is determined by how quantities like the energies and angular momenta are distributed. For practical purposes we may take 30 crossing times as a maximum in order to reach this equilibrium. In the remainder of this chapter we will limit ourselves strictly to equilibrium configurations.

2.2 The Jeans theorem

Let us first make use of the fact that we are neglecting close encounters by eliminating the stellar masses from the distribution function. We define the **reduced distribution function** as:

$$\varphi = \int_0^\infty \Psi m \, dm$$

The density equation can now be written:

$$\rho = \int\int\int_{-\infty}^{+\infty} \varphi \, dudvdw$$

and the Poisson equation – Eq. (21) – remains unchanged. The Liouville equation can be multiplied by $m\,dm$ and integrated from 0 to ∞, yielding the same equation featuring φ instead of Ψ:

$$\frac{\partial\varphi}{\partial t} + u\frac{\partial\varphi}{\partial x} + v\frac{\partial\varphi}{\partial y} + w\frac{\partial\varphi}{\partial z} - \frac{\partial U}{\partial x}\frac{\partial\varphi}{\partial u} - \frac{\partial U}{\partial y}\frac{\partial\varphi}{\partial v} - \frac{\partial U}{\partial z}\frac{\partial\varphi}{\partial w} = 0 \qquad (29)$$

Note that φ physically means the mass density in a 6-dimensional reduced phase space (x, y, z, u, v, w). Also note that getting rid of the masses was easy with the collision-free Boltzmann equation, but it would have been impossible, if the 'collision term' due to close encounters had been retained in the right-hand member. This term would have featured the stellar masses in a complicated way. A curious fact is that, by eliminating the masses in a collision-free system as above, we have shown that it does not matter at all, how the mass is distributed between different stars, as long as the total mass per unit volume is conserved. The dynamics is the same, whether we deal with stars or planets or dust grains!

Now return to the equations governing the motion of a star:

$$\frac{dx}{dt} = u\,; \ldots\,; \frac{du}{dt} = -\frac{\partial U}{\partial x}\,; \ldots$$

and assume that $U(x, y, z, t)$ is known. This is a sixth order system of differential equations.

Let us repeat a few mathematical concepts concerning such a system. An **integral** of the system is a function $I(x, y, z, u, v, w, t)$, which has a constant value, when inserting an arbitrary solution $x(t), \ldots, w(t)$ of the system. Several integrals $I_1, I_2, \ldots, I_n$ are called **independent**, if there is *no* relation $g(I_1, I_2, \ldots, I_n) \equiv 0$ between them. For a system of order γ one may find γ independent integrals, but no more. In the present case we have $\gamma = 6$ and thus there are six independent integrals $I_1, \ldots, I_6$.

Now let I be any arbitrary integral. The seven integrals $I_1, \ldots, I_6, I$ cannot be independent, and thus there is a relation:

$$g(I_1, \ldots, I_6, I) \equiv 0$$

which can be solved for I:

$$I = f(I_1, \ldots, I_6) \tag{30}$$

Hence every integral is a function of the six independent integrals. And conversely, every such function is an integral – since each of the arguments is constant, the function is constant too. Therefore, Eq. (30) is the general expression for all integrals of the system, if f is an arbitrary function. We see that it suffices to find six independent integrals in order to know them all.

We have by definition: $I[x(t), \ldots, w(t), t] \equiv$ *const.*, if $x(t)$ etc. describe the motion of any star. If we differentiate, we obtain:

$$\frac{\partial I}{\partial x}\frac{dx}{dt} + \cdots + \frac{\partial I}{\partial w}\frac{dw}{dt} + \frac{\partial I}{\partial t} = 0$$

or, using the equations of motion:

$$\frac{\partial I}{\partial x}u + \cdots - \frac{\partial I}{\partial u}\frac{\partial U}{\partial x} - \cdots + \frac{\partial I}{\partial t} = 0$$

But this is exactly the Liouville equation, if we replace I by φ. In other words: **The distribution function is an integral of the equations of motion**. Of course, we had already seen that φ is conserved, when following the motion, so indeed it has to be an integral.

Summarizing:

The general solution of the Liouville equation is:

$$\boxed{\varphi = f(I_1, I_2, \ldots, I_6)} \tag{31}$$

where the I's are six independent integrals of the equations of motion and f is an arbitrary function. This result is known as the **Jeans theorem**.

Note that in the formal expressions for I, we have not constrained them to be explicitly independent of time. However, we are interested in stellar systems that have reached equilibrium and are in stationary states. Hence we consider systems for which $\partial\varphi/\partial t = 0$. In this case ρ

and U are also independent of time, and we shall show that the six independent integrals can be chosen in such a way that only one of them depends explicitly on time.

Proof: We can eliminate dt between the equations of motion:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{du}{-\frac{\partial U}{\partial x}} = \frac{dv}{-\frac{\partial U}{\partial y}} = \frac{dw}{-\frac{\partial U}{\partial z}}$$

Since U does not depend on time, this system of equations is independent of time. These are the *equations of the trajectory* – *i.e.*, they determine the curves followed by the representative points in phase space. We may take any of the variables, *e.g.*, x, as independent variable and the others as dependent, so we can write the system:

$$\frac{dy}{dx} = \frac{v}{u}; \ \frac{dz}{dx} = \frac{w}{u}; \ldots$$

and we see that this system is of *fifth order*.

Hence there are five independent integrals, *i.e.*, quantities $I_i(x, y, z, u, v, w)$ that remain constant along any trajectory. But a quantity that is explicitly independent of time and constant along a trajectory is obviously also constant, when following the motion – thus, these five time-independent integrals are also integrals of the equations of motion. *Q.E.D.*

Among the six integrals we thus distinguish five $(I_1, \ldots, I_5)$ that are explicitly independent of time and are called **conservative integrals**, and one integral (I_6) that does depend on time and is called a **non-conservative integral**. But now we return to the requirement that φ should be independent of time. A necessary and sufficient condition for this is that φ does not depend on I_6, since only I_6 depends on time. Thus we have the following result:

In a stationary system the general form of the distribution function is:

$$\boxed{\varphi = f(I_1, I_2, \ldots, I_5)} \tag{32}$$

where the I's are five independent, conservative integrals of the equations of motion, and f is an arbitrary function. This specialized version of the Jeans theorem is the one that is most used in practice.

In terms of geometry, consider a conservative integral $I(x, \ldots, w)$. The equation $I = c$ (a constant) thus defines a 5-dimensional **hyperplane** within the 6-dimensional phase space. If we let c take all possible values, we get a *family* of hyperplanes, which fills up the whole phase space. If we have γ equations: $I_1(x, \ldots, w) = c_1$, …, $I_\gamma(x, \ldots, w) = c_\gamma$ – each one defining a hyperplane, then the intersection of these hyperplanes becomes a subspace of $6 - \gamma$ dimensions. Specifically, if $\gamma = 5$, we have a one-dimensional subspace, *i.e.*, a curve.

Now, consider a trajectory. Each one of the independent, conservative integrals has a constant value along this trajectory. We thus have: $I_1 = c_1$, …, $I_5 = c_5$ at all points on the trajectory. We conclude that *the trajectory is part of the intersection of the five hyperplanes* $I_1 = c_1$, …, $I_5 = c_5$. We say 'part of', because the trajectory is one curve, but the intersection may consist of two or more separate curves.

2.2.1 Isolating and non-isolating integrals

We shall now see that there is no equality between conservative integrals. They are actually of two kinds with very different properties, called **isolating integrals** and **non-isolating integrals**. We will make a simple illustration using a system moving in two dimensions (x, y).

The phase space thus has four dimensions, and we have four independent integrals of the equations of motion, whereof three are conservative. Let us take a simple form for the potential:

$$U = \frac{1}{2}\left(a^2x^2 + b^2y^2\right)$$

where a and b are constants. The equations of motion are:

$$\begin{aligned} \dot{x} &= u & \dot{y} &= v \\ \dot{u} &= -a^2x & \dot{v} &= -b^2y \end{aligned}$$

These are easily integrated, and the general solution is:

$$\begin{aligned} x &= x_o \sin a(t - t_1) \\ y &= y_o \sin b(t - t_2) \\ u &= a x_o \cos a(t - t_1) \\ v &= b y_o \cos b(t - t_2) \end{aligned}$$

where x_o, y_o, t_1 and t_2 are four integration constants. We easily find four independent integrals:

$$\begin{aligned} I_1 &= x_o = \pm\sqrt{x^2 + u^2/a^2} \\ I_2 &= y_o = \pm\sqrt{y^2 + v^2/b^2} \\ I_3 &= t_1 = t - \frac{1}{a}\arctan(ax/u) \\ I_4 &= t_2 = t - \frac{1}{b}\arctan(by/v) \end{aligned}$$

These are functions of x, y, u, v, t. Note that there is a close connection between *integrals* and *integration constants*.

Only two of the above integrals are conservative, but we can easily produce a third one from them:

$$I_3' = I_4 - I_3 = \frac{1}{a}\arctan(ax/u) - \frac{1}{b}\arctan(by/v)$$

The Jeans theorem tells us that the general form of the distribution function is $\varphi = f(I_1, I_2, I_3')$, f being an arbitrary function.

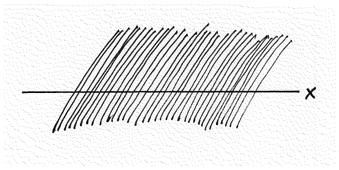
But when scrutinizing the integral I_3', we find that it has rather peculiar properties. Recall that each arctan function has an infinity of values, being defined only disregarding multiples $k\pi$ (k is an arbitrary integer). If we thus let $I_{3,0}'$ be one special value of I_3', the others are given by:

$$I_3' = I_{3,0}' + \frac{k\pi}{a} + \frac{\ell\pi}{b}$$

where k and ℓ are arbitrary integers.

At any given point in phase space, I_3' thus has an infinity of values. If the ratio a/b is irrational, the set of these values is everywhere compact, *i.e.*, in whatever interval, no matter how small, we can always find values of I_3'. Conversely, consider the equation $I_3' = c_3'$ (a constant). In principle, this defines a 3-dimensional hyperplane in the 4-dimensional phase space. But if we solve this equation for one of the variables, *e.g.*, x, we get:

$$x = \frac{u}{a}\tan\left[a\left(c_3' + \frac{1}{b}\arctan(by/v)\right)\right]$$



and – once again – since the arctan function has an infinity of values, and if b/a is irrational (which is the general case), we get an infinity of values of x, which is everywhere compact. This means, by a 2-dimensional analogy, that the hyperplane consists of an infinite set of "sheets" situated infinitely close to each other, somewhat like the pages of a book. These sheets fill up the whole phase space. Thus, in practice, the subspace $I'_3 = c'_3$ coincides with phase space, *i.e.*, the condition $I'_3 = c'_3$ is like no condition at all. We call the integral I'_3 **non-isolating**.

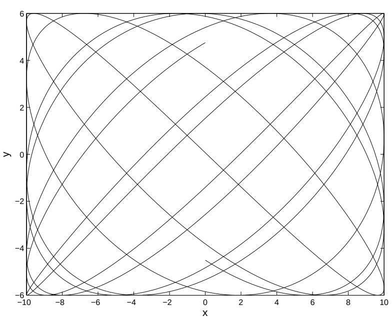
Definition: **An integral I is called non-isolating, if the set of points satisfying $I = c$ is everywhere compact in phase space.**

On the other hand, the other integrals of the above example define "normal" hyperplanes consisting of separate sheets. For instance, $I_1 = c_1$ gives:

$$x = \pm\sqrt{c_1^2 - a^2u^2}$$

i.e., two sheets. These are **isolating** integrals.

We have seen that a trajectory is part of the intersection of the three hyperplanes $I_1 = c_1$; $I_2 = c_2$; $I'_3 = c'_3$. In principle, this intersection should be a curve. But since I'_3 is non-isolating, we have in practice only two hyperplanes. Their intersection is a subspace with dimension $4 - 2 = 2$, *i.e.*, a surface.



Will the trajectory cover this surface? Yes, it will, as demonstrated by the orbit plotted to the left for one particular case. This orbit arises from the combination of a sinusoidal movement in x and one in y with different periods – a **Lissajous figure**. If b/a is irrational, it will eventually fill the whole rectangle. But the orbit is actually a *projection* of the trajectory onto the (x, y) plane, so if it fills up a surface, this must be because the trajectory also fills up a surface.

Hence, in practice, everything appears as if the non-isolating integral did not exist. In fact, one can easily prove that the distribution function φ, which according to the previous results should have the general form for a stationary state: $\varphi = f(I_1, I_2, I'_3)$, in fact cannot depend on I'_3.

One might think that we have come across a very special case and that the non-isolating integrals are exceptional in occurrence. But this is not so, as we shall see later on – by contrast, they appear in almost all situations. Thus we have the following, final version of the Jeans theorem:

In a stationary state, the general form of the distribution function is $\varphi = f(I_1, \ldots, I_\gamma)$, where the I's are the independent, conservative and isolating integrals of the equations of motion.

2.2.2 The energy integral

From what we have just seen, we are forced to look for conservative and isolating integrals in order to build models of stationary systems, *i.e.*, select appropriate distribution functions. But, alas, there is only one such integral that is generally known! This is the total energy of a star per unit mass:

$$\mathcal{E} = \frac{1}{2}\left(u^2 + v^2 + w^2\right) + U(x, y, z) \tag{33}$$

Note that the system is stationary, and therefore the time does not appear in the potential energy term. We easily verify that this is an integral. The requirement is:

$$u\frac{\partial \mathcal{E}}{\partial x} + \cdots - \frac{\partial U}{\partial x}\frac{\partial \mathcal{E}}{\partial u} - \cdots = 0$$

i.e., using Eq. (33),

$$u\frac{\partial U}{\partial x} + \cdots - \frac{\partial U}{\partial x}u - \cdots = 0$$

which is obviously satisfied. As we easily see, this integral is both conservative and isolating.

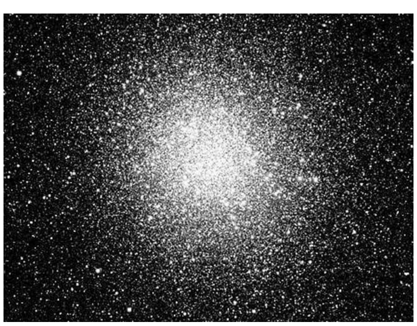
In the general case, unfortunately, we know next to nothing about the remaining four conservative integrals. Long ago a hypothesis was advanced, according to which these integrals are in general non-isolating – this is the **ergodic hypothesis**, borrowed from the statistical theory of gases. But in fact it cannot be true, at least not for an isolated system. To see this, suppose that there is no other conservative, isolating integral than $\mathcal{E}$. Then we have necessarily: $\varphi = f(\mathcal{E})$. But in such a case it can be shown from the symmetric appearance of the velocity components in $\mathcal{E}$ that the position coordinates have to appear symmetrically too, so the system has spherical symmetry. However, we shall soon see that such systems have three more isolating integrals, so we have reached a contradiction.

In the following we will investigate a number of cases, where the potential has some special, symmetric form and which seem to be of practical interest for real stellar systems.

In fact, many systems exhibit a rather striking **spherical symmetry**. This holds for globular clusters as well as old open clusters, and to some extent for galaxy clusters. Even more general is the occurrence of **rotational symmetry**, of which spherical symmetry is a special case. The fact that most real systems exhibit rotational symmetry can be intuitively understood as a result of dynamical mixing, which tends to reduce any irregularities and give maximum symmetry to the system. But in the presence of rotation, and thus a conserved total angular momentum, this vector will enforce a privileged direction to the system. The maximum symmetry that is consistent with such a privileged direction is rotational symmetry.

But this is not a proof. In fact, it has been found that a stationary system does not necessarily possess any axis of symmetry. Triaxial models can be constructed for elliptical galaxies in a stationary state, and real galaxies may also have such shapes.

2.3 Spherical symmetry



Globular clusters are prime examples of stellar systems with spherical symmetry. In reality, of course, nothing is perfect, and we do not need to take for granted that all globulars are perfectly spherical in shape. But dynamical models that assume spherical symmetry will at least be good approximations, and this Section will deal with the construction of such models. The picture to the left shows *Messier 22* as a typical example of a globular cluster in our Galaxy.

2.3.1 Integrals

We assume that all the basic quantities have spherical symmetry about the origin, which is of course at the center of the system. For instance, the density – generally an arbitrary function of x, y, z – shall now be constant on each sphere centered on the origin, *i.e.*, we have $\rho = \rho(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin. Likewise with the potential: $U = U(r)$. Thus the force acting on a star is always directed toward the origin. By consequence, the angular momentum about the origin per unit mass is constant, since we have:

$$\mathbf{A} = \mathbf{r} \times \dot{\mathbf{r}}$$

and thus:

$$\frac{d\mathbf{A}}{dt} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{0}$$

The three components of the vector $\mathbf{A}$ hence remain constant in the course of the motion and provide three integrals. So, in addition to the energy integral $I_1 = \mathcal{E}$ as given by Eq. (33), we have:

$$\begin{aligned} I_2 &= A_x = & yw - zv \\ I_3 &= A_y = & zu - xw \\ I_4 &= A_z = & xv - yu \end{aligned}$$

We easily verify that these quantities are integrals. For instance:

$$u\frac{\partial A_x}{\partial x} + \cdots - \frac{\partial U}{\partial x}\frac{\partial A_x}{\partial u} - \cdots = 0$$

can be written:

$$vw - wv + \frac{\partial U}{\partial y}z - \frac{\partial U}{\partial z}y = 0$$

But we have:

$$\frac{\partial U}{\partial y} = \frac{dU}{dr} \cdot \frac{\partial r}{\partial y} = \frac{dU}{dr} \cdot \frac{y}{r}$$

and likewise for $\partial U/\partial z$, so the equality is obvious.

These integrals are seen to be conservative and isolating. One can also verify that they are independent internally as well as of $\mathcal{E}$. Thus we know all conservative integrals except I_5. We shall see that this is in general non-isolating. Therefore we can write the reduced distribution function:

$$\varphi = f(\mathcal{E}, A_x, A_y, A_z)$$

But the spherical symmetry tells us that φ must be the same for any given absolute value of the angular momentum independent of direction. In other words, φ may only depend on the absolute value A of the angular momentum:

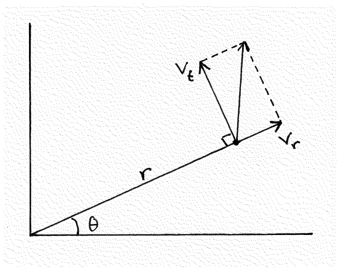
$$A = |\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Hence:

$$\varphi = f(\mathcal{E}, A) \tag{34}$$

2.3.2 Stellar orbits

The vector $\mathbf{A}$ is always perpendicular to $\mathbf{r}$. Therefore, $\mathbf{A} = \mathbf{const}$. implies that the motion stays in a plane orthogonal to $\mathbf{A}$. So we conclude that stellar orbits are planar.



Let us take a polar coordinate frame in the plane of the orbit. The velocity vector has two components:

radial velocity	$v_r = \dot{r}$
transverse velocity	$v_t = r\dot{\theta}$

The energy integral can be written in polar coordinates:

$$\mathcal{E} = U(r) + \frac{1}{2}\left(v_r^2 + v_t^2\right) = U(r) + \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{35}$$

and the absolute value of the angular momentum:

$$A = rv_t = r^2\dot{\theta} \tag{36}$$

We can solve for $\dot{\theta}$ and $\dot{r}$ from these equations:

$$\begin{cases} \dot{\theta} = A/r^2 \\ \dot{r} = \pm\sqrt{2\mathcal{E} - 2U(r) - A^2/r^2} \end{cases}$$

Concerning integrals of motion, two of them correspond to the angles defining the plane of the orbit. $\mathcal{E}$ and A are two of the four remaining ones, and we can now easily integrate the above equations to find the last two. We start by the second one, which only features r and t:

$$t = t_o + \int \frac{dr}{\pm\sqrt{2\mathcal{E} - 2U(r) - A^2/r^2}} \tag{37}$$

Then we can divide the first equation by the second, thus eliminating dt and obtaining an equation for the dependence between θ and r. Integrating this, we get:

$$\theta = \theta_o + A \int \frac{dr}{\pm r^2\sqrt{2\mathcal{E} - 2U(r) - A^2/r^2}} \tag{38}$$

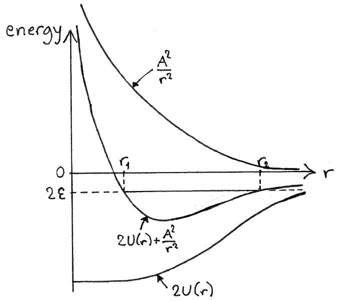
This completes the solution of the problem, and the last two integrals are the new constants of integration: t_o and θ_o. We see from Eq. (38) that θ_o is conservative, while Eq. (37) shows that t_o is not. Both of them can be given the value zero by a proper choice of the origin of time and of angle in the orbital plane, but we are not free to assign special values to $\mathcal{E}$ and A in such a manner. In fact, they determine the size and shape of the orbit.

The equation for $\dot{r}$ shows that we must have:

$$2\mathcal{E} - 2U(r) - \frac{A^2}{r^2} \geq 0$$

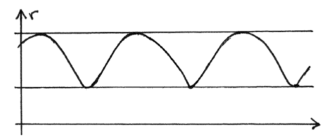
i.e.,

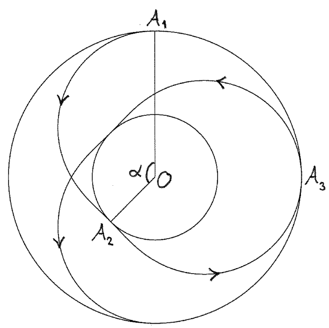
$$2U(r) + \frac{A^2}{r^2} \leq 2\mathcal{E} \tag{39}$$



The smoothed potential function should look roughly as shown to the left, with a minimum at the center and slowly rising toward zero as the distance grows toward infinity. Thus, as concerns the curve $f(r) = 2U(r)+A^2/r^2$, the second term will dominate close to the origin, but the first term will dominate at very large distances, where we can expect $U(r) \propto -1/r$. Somewhere in between there is a minimum. This is the general behaviour of the curve.

If we assume a given value of A, and thus a given curve as shown above, the value of $\mathcal{E}$ determines the limits of the orbit. For the value indicated, we see that the condition of Eq. (39) is satisfied only for distances from r_1 to r_2. At these special values, the radial velocity vanishes. We easily realize that the variation of r with time is a *periodic oscillation* between the extreme values r_1 and r_2.

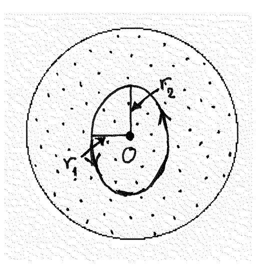




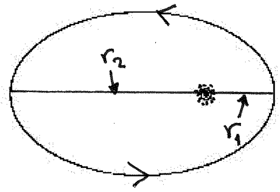
If we combine this oscillation with a steady rotation in θ, as given by Eq. (38), we obtain the orbit. This has the general form of a *rosette*, drawn within a circular annulus with radii r_1 and r_2. One can show that the orbital segments $\mathcal{A}_2\mathcal{A}_3$ and $\mathcal{A}_1\mathcal{A}_2$ are symmetric with respect to the line $\mathcal{A}_2\mathcal{O}$, etc.

The angle α ($\mathcal{A}_1\mathcal{O}\mathcal{A}_2$) in general does not take an even value, but it can be shown to be in the interval $[\pi/2, \pi]$. If the ratio α/π is rational, the orbit will close upon itself (*e.g.*, in the case indicated in the above figure, where $\alpha = 3\pi/4$). But in general this ratio is irrational, so the orbit never repeats and gradually fills up the whole circular annulus. This corresponds to the existence of a non-isolating integral, as we have already foreseen.

In fact there are only two shapes of the potential $U(r)$, for which all orbits are closed, namely, the following ones.



• **Homogeneous system:** $\rho(r) = const$, from which we may derive: $U(r) = U(0) + ar^2$ with $a = const$. In this case the orbits are ellipses with $\mathcal{O}$ at their centres, and $\alpha = \pi/2$ for all these orbits.



• **Point mass system:** The whole mass is gathered in the centre, so $\rho(r) = \delta(r)$, where δ is the Dirac delta function. Thus $U(r) = -GM/r$ for the two-body problem. All the orbits are ellipses with $\mathcal{O}$ at one of their foci, and $\alpha = \pi$.

These two cases do not occur in reality, because in real stellar systems the mass is distributed in space, and the density decreases outward. However, the first case may serve as an approximation for the *central part* of a stellar cluster, *i.e.*, for stars moving at small r, and the second may serve as an approximation for stars moving at large r in the *outskirts* of the cluster. It appears that in real stellar clusters α should vary continuously from $\pi/2$ near the center to π for distant orbits.

2.3.3 Plummer model

We have seen that $\varphi = f(\mathcal{E}, A)$. Let us suppose that f is *given* with some arbitrary expression, and then try to derive ρ and U. We have:

$$\varphi = f\left[U(r) + \frac{v_r^2 + v_t^2}{2},\ rv_t\right]$$

If $U(r)$ were known, φ would be a known function of r, v_r and v_t. But since this is not the case, we instead write:

$$\varphi = g(U,\ r,\ v_r,\ v_t)$$

with the intention to substitute $U = U(r)$ later on. As f is given, g is also given. We then obtain the density from:

$$\rho = \int\int\int \varphi\, dudvdw = 2\pi \int\int g\, dv_r dv_t$$

so we can write:

$$\rho = h(U, r)$$

where h can also be computed from the given expression for f. We insert this into the Poisson equation, which due to spherical symmetry may be written:

$$\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} = 4\pi G h(U, r) \tag{40}$$

This is an ordinary differential equation for $U(r)$, which may be solved. Due to the boundary condition $U(\infty) = 0$, the solution is generally unique. If we substitute this solution into g and h, we thus obtain $\varphi(r, v_r, v_t)$ as well as $\rho(r)$. We then have a model satisfying all the fundamental equations.

We started out with an arbitrary function f, so in a sense there are as many models for a spherical system as there are functions of two variables! However, some functions are forbidden because f has to be non-negative everywhere, and some are unphysical since f should be zero for positive energies (such stars would quickly escape from the system). In spite of this, a large number of models (some even unphysical) have been proposed by different authors. We shall only dwell upon the **Plummer model**, which is used very frequently and was historically the first. In this case our *ansatz* is:

$$\boxed{\varphi = \begin{cases} a(-\mathcal{E})^{7/2}; & \mathcal{E} < 0 \\ 0\ ; & \mathcal{E} \geq 0 \end{cases}} \tag{41}$$

where a is a positive constant. Note that this model assumes a distribution function that does not depend on angular momentum but only on energy. In this case the g function has a simple expression:

$$\varphi = g(U, V) = a(-U - V^2/2)^{7/2}$$

where V is the absolute value of the total velocity, which is limited to values less than V_ℓ in order to avoid positive energies. We get:

$$\rho = \int\int\int \varphi \, du dv dw = 4\pi \int \varphi V^2 \, dV$$

using spherical coordinates in velocity space, and this can be expressed:

$$\rho = 4\pi \int_0^{V_\ell} a \left(-U - \frac{V^2}{2} \right)^{7/2} V^2 \, dV$$

with

$$V_\ell = \begin{cases} \sqrt{-2U}; & U < 0 \\ 0 \quad ; & U \geq 0 \end{cases}$$

We put:

$$V = \tau \sqrt{-U} \quad \Rightarrow \quad \rho = 4\pi a (-U)^5 \int_0^{\sqrt{2}} \left(1 - \frac{\tau^2}{2} \right)^{7/2} \tau^2 \, d\tau$$

The integral has a definite value, and thus: $\rho = C(-U)^5$. The Poisson equation then yields:

$$\frac{d^2 U}{dr^2} + \frac{2}{r} \frac{dU}{dr} = \begin{cases} 4\pi G C(-U)^5; & U < 0 \\ 0 \quad ; & U \geq 0 \end{cases}$$

This equation is known also from the theory of the interior structure of stars. The solution is called a **polytrope** of index 5. It is expressed analytically:

$$U = \frac{U_o}{(1 + r^2/r_o^2)^{1/2}}$$

where U_o and r_o are constants, which yields:

$$\boxed{\rho = \frac{\rho_o}{(1 + r^2/r_o^2)^{5/2}}} \qquad (42)$$

with

$$\rho_o = const. = -\frac{3U_o}{4\pi G r_o^2}$$

2.3.4 Other models

The Plummer model is characterized by the simple power-law dependences as described above and provides a reasonable fit to the observed structure of stellar clusters. But we also need to mention another category of models, which has a completely different physical background. Generally speaking, we can call these **isothermal models**. We will come to the explanation in later Sections, but the underlying idea is that many clusters may be *collisionally evolved* or *thermalized*, so that a multitude of close encounters between stars have set up a characteristic energy distribution reminiscent of the *Boltzmann distribution* of kinetic gas theory. Thus the repartition of energies is like that of an equilibrium gas and can be described by means of a unique *temperature*.

In the simplest of such models one puts:

$$\varphi = a \cdot e^{-b\mathcal{E}} \qquad (43)$$

with two constants a and b. Since this yields

$$\varphi = a \cdot e^{-bU} \cdot e^{-bV^2/2},$$

we see that the velocity distribution is Maxwellian everywhere with a constant dispersion. This shows the analogy with the molecules of a gas and that the 'temperature' is constant everywhere – thus the name of this kind of model.

However, it can be shown that the above assumption leads to unrealistic consequences. In particular, the potential function derived from Eq. (43) diverges for $r \to \infty$ and the total mass becomes infinite. The reason is that we have allowed not only negative energies but positive ones as well, whilst in a real system the size is limited amd the stars with positive energies have already escaped and are no longer part of the system.

There are several kinds of modifications in common use, where this inconvenience has been removed by cutting the energy distribution such that positive energies are excluded. A typical example is the *King model* introduced by the stellar dynamicist Ivan King, which uses:

$$\boxed{\varphi = a\left\{e^{b(\mathcal{E}_t-\mathcal{E})} - 1\right\}; \quad \mathcal{E} < \mathcal{E}_t} \tag{44}$$

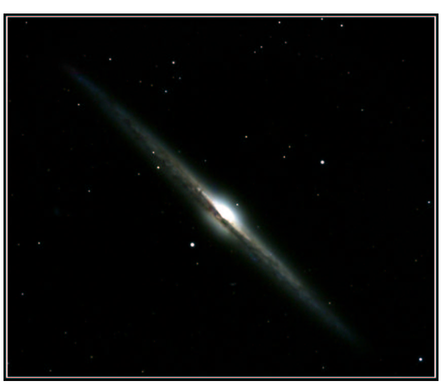
Here $\mathcal{E}_t$ is a "tidal energy", corresponding to a *tidal radius* of the cluster. As we shall see later, this comes from the fact that the tidal force of the Galactic potential, into which a stellar cluster is immersed, sets a limit to the size of the cluster. Stars that venture too far away get stripped away from the cluster by this tidal force.

Another kind of model building on the isothermal assumption is called the *Eddington model* after Arthur Eddington. Here the modification consists in accounting also for the *angular momentum* integral by writing:

$$\boxed{\varphi = a \cdot e^{-b\mathcal{E}-cA^2}} \tag{45}$$

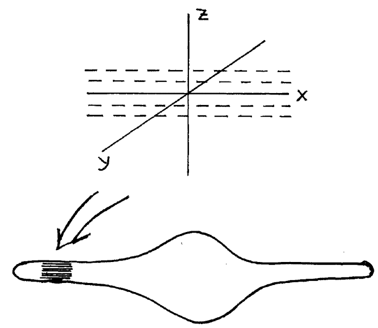
This is unrealistic in the same way as the above isothermal model, so one may modify it similar to what was done in the King model. But the essential property of the Eddington model is that the velocity distribution is no longer isotropic (like in a gas). It can easily be shown that, for positive values of b and c, the Gaussian distribution of the radial component (*i.e.*, velocities in the inward or outward directions with respect to the center of the cluster) is larger than those of the transverse components. The reason for such an asymmetry could be due to the way the cluster was formed, *e.g.*, by radial collapse of a parent gas cloud.

2.4 Plane-parallel systems



The picture to the left shows the disk galaxy NGC 4565, seen edge-on. Such systems exhibit another kind of symmetry, where there seems to be a central plane, and the distribution of stars seems symmetric with respect to that plane. Furthermore, considering a local region of the disk, the number density of stars does not seem to vary along the disk but only perpendicular to it. This means that we have plane-parallel geometry. Our own Galaxy is of course a disk galaxy too, and one often uses this plane-parallel geometry to study the dynamics of stellar motions perpendicular to the Galactic disk in the solar neighbourhood.

2.4.1 Integrals



We assume that φ, ρ and U depend only on z but not on x or y:

$$\rho = \rho(z) \quad ; \quad U = U(z)$$

and for a stationary system:

$$\varphi = \varphi(z, u, v, w)$$

The density and potential are constants on each horizontal plane.

Let us integrate the distribution function over u and v:

$$\varphi_1(z, w) = \int\int \varphi \, du \, dv$$

We then have:

$$\rho = \int \varphi_1 \, dw$$

The Poisson equation is written:

$$\nabla^2 U = \frac{\partial^2 U}{\partial z^2} = 4\pi G \rho$$

and the Liouville equation reduces to:

$$w \frac{\partial \varphi_1}{\partial z} - \frac{\partial U}{\partial z} \frac{\partial \varphi_1}{\partial w} = 0$$

Thus, by eliminating the variations in the x and y directions, we have got a one-dimensional motion with a two-dimensional phase space. There are two independent integrals, whereof one is conservative. But we already know this integral, *i.e.*, the *energy integral*, whose one-dimensional appearance is:

$$\mathcal{E} = U(z) + \frac{w^2}{2}$$

Hence we immediately have the general solution:

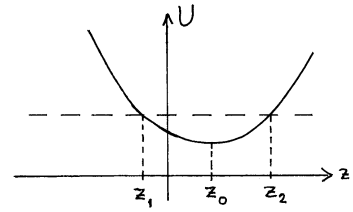
$$\varphi_1 = f(\mathcal{E}) \tag{46}$$

where f is an arbitrary function. There are no non-isolating integrals.

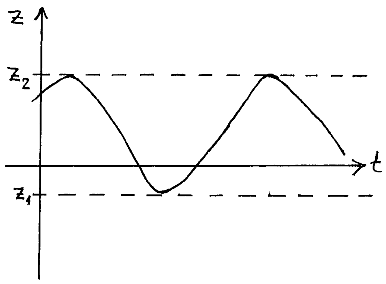
2.4.2 Stellar orbits

Since $U = U(z)$, the force acting on a star is vertical. Thus the horizontal velocity components u and v are constants – the projection of the motion onto the (x, y) plane is uniform and rectilinear. The vertical motion is what interests us, and this is given by:

$$\dot{z} = w \quad ; \quad \dot{w} = -\frac{\partial U}{\partial z} \quad \text{or :} \quad \ddot{z} = -\frac{\partial U}{\partial z}$$



The Poisson equation shows that $U(z)$ is a concave function; it has a unique minimum at $z = z_o$. Thus we have a *restoring force* toward the plane $z = z_o$, and the motion consists of oscillations around this plane. In reality, of course, z_o should correspond to the central plane of the disk, and we can take it to be zero.



To integrate the motion, we make use of the energy equation, which yields:

$$\dot{z} = \pm\sqrt{2\mathcal{E} - 2U(z)}$$

and hence:

$$t = t_o + \int \frac{dz}{\pm\sqrt{2\mathcal{E} - 2U(z)}}$$

providing the relation between z and t. $\mathcal{E}$ and t_o are the two integration constants. We must have: $U < \mathcal{E}$, and hence z is confined between the two extrema z_1 and z_2. For these values the vertical velocity vanishes. The motion is thus a periodic oscillation between z_1 and z_2.

2.4.3 An isothermal model

As an example of model construction in plane-parallel geometry, let us take the isothermal model that we mentioned in the spherically symmetric case. We apply Eq. (43), this time considering only the vertical motion, and we get:

$$\varphi_1 = a \cdot e^{-bU} \cdot e^{-bw^2/2}$$

Next we integrate this to derive the relation between density and potential:

$$\rho = a\, e^{-bU} \int_{-\infty}^{+\infty} e^{-bw^2/2}\, dw = a\sqrt{\frac{2\pi}{b}} e^{-bU}$$

and inserting this into the Poisson equation, we get:

$$\frac{\partial^2 U}{\partial z^2} = 4\pi G a \sqrt{\frac{2\pi}{b}} e^{-bU}$$

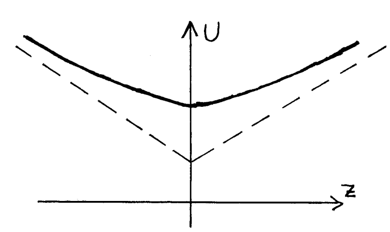
The general solution of this differential equation is:

$$U = U_o + \frac{2}{b} \ln \coth \frac{z - z_o}{h} \qquad (47)$$

where z_o and h are two integration constants, and U_o is a third constant, related to h by:

$$U_o = \frac{1}{2b} \ln\left(8\pi^3 G^2 a^2 b h^4\right) \qquad (48)$$

We can take $z_o = 0$ without loss of generality, and the solution for $U(z)$ has the shape indicated to the right. We have $U(0) = U_o$, and the function is symmetric so that $U(-z) = U(z)$ everywhere. For $z \to \infty$ the hyperbolic cotangent function is approximated by $\coth(z/h) \simeq e^{z/h}/2$, so that:



$$U \simeq U_o + \frac{2}{b} \ln \frac{1}{2} + \frac{2}{bh} z$$

which describes the asymptotic behaviour seen in the figure. Note that this is the same problem that was mentioned above for the spherically symmetric case – the isothermal system has an infinite extent and a diverging gravitational potential. But in reality this is no problem in the present case, because the plane-parallel approximation can only be used for small distances from the central plane, *i.e.*, small values of $|z|$. We have to imagine that the real potential function flattens out at large $|z|$, but this happens outside the region that we are interested in. The density function obtained by using Eqs. (47) and (48) is:

$$\rho = \frac{1}{2\pi G b h^2} \cdot \frac{1}{\coth^2(z/h)}, \tag{49}$$

and this is a quasi-Gaussian function with a relatively flat central peak and a very rapid decrease at large distances.

The vertical acceleration $-\partial U/\partial z$ vanishes in the central plane and is approximately proportional to $-z$ at small distances. Thus the vertical motions with small amplitude are practically sinusoidal oscillations.

2.4.4 The local Galactic disk

The above-described isothermal model is a reasonable approximation to the local Galactic disk, but it does not provide all the answers. In particular one needs to determine the *scale height* h and the actual midplane density ρ_o. This has to be done using observations of stars. However, the practical work involved complications that will now be described.

First of all, the census of nearby stars is not at all complete, and even if it were, one would also have to include other possible contributors, like interstellar gas or even dark matter, which of course we can not observe. The solution is to use a particular kind of stars as a *tracer population*. These should be relatively common so as to be observable in large numbers, and also relatively luminous in order to be observable at fairly large distances. Typical examples may be F or K giants.

Observing the closest stars, *e.g.*, the closest K giants, we can determine their distribution of w velocities, and since the Sun is situated just a few parsecs from the Galactic midplane, we can assume $z = 0$ in this case. Thus we get the local distribution function $\varphi_{1K}(0, w)$, from which we derive the function $f_K(\mathcal{E})$. Then we use the following relation for the density:

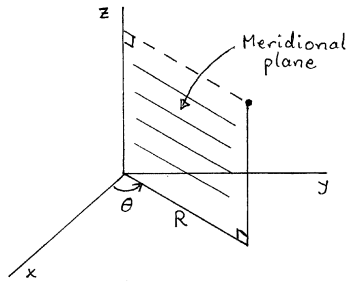
$$\rho_K(z) = \int f_K \left[U + \frac{w^2}{2} \right] dw = h_K(U) \tag{50}$$

The above integral can be evaluated for all values of U. Thus we determine the function h_K, *i.e.*, the relation between ρ_K and U. We now use observations of $\rho_K(z)$, *i.e.*, the fall-off of the number density of K giants with $|z|$, and from this we derive $U(z)$, *i.e.*, the total gravitational potential including all contributions. Finally, we may differentiate $U(z)$ twice and, using the Poisson equation, derive the variation of the total density $\rho(z)$. This includes the local density ρ_o.

This kind of analysis was pioneered in the 1930's by Jan Oort, but even relatively recently the observational material was not of sufficient quality to obtain accurate results. An example is an in-depth study by John Bahcall in 1984, which gave $\rho_o = 0.185 \pm 0.02\ M_\odot/\mathrm{pc}^3$. By adding up the mass of observed stars and interstellar material, one could only account for part of this density, which led to the suspicion of a substantial amount of dark matter in the Galactic disk. However, after the *Hipparcos* satellite led to a great improvement of the knowledge of stellar distances and velocities in the 1990's, a new analysis by Johan Holmberg and Chris Flynn in 2000 instead yielded $\rho_o = 0.102 \pm 0.01\ M_\odot/\mathrm{pc}^3$, thus effectively removing the need for any dark matter in the local disk.

2.5 Axial symmetry

Now assume that the system is axially symmetric, *i.e.*, that the distribution function is invariable under rotation around a given axis.



Let us take a *cylindrical coordinate frame* (R, θ, z) as sketched in the figure to the left. The density and potential are hence functions $\rho(R, z)$ and $U(R, z)$, and the *meridional plane* is the plane that passes through the given point and the z axis. An immediate consequence of the axial symmetry of U is that the force acting on any star is confined to the meridional plane, but it does not have to point toward the origin, and in general it does not. Hence the torque is perpendicular to the meridional plane and in particular the z axis, so the angular momentum component along the z axis (the z-component) is conserved. Thus we have an integral:

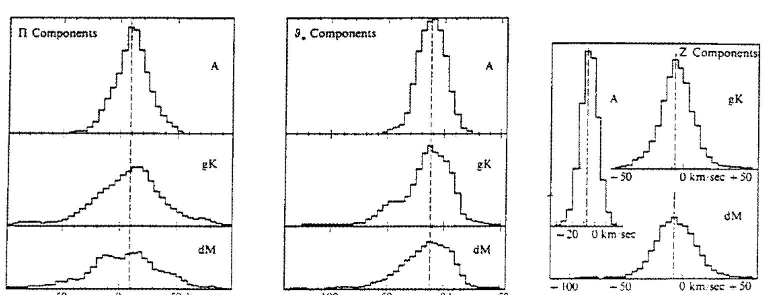
$$A_z = xv - yu = R^2 \dot{\theta} \tag{51}$$

2.5.1 The third integral

Apart from $\mathcal{E}$ and A_z, no other integrals are known in the general case. A lot of effort has been devoted to the question whether all three remaining conservative integrals are non-isolating, or if one of them – usually called **the third integral** – may in fact be isolating. One important piece of evidence has come from the **velocity ellipsoid**, *i.e.*, an ellipsoidal figure in velocity space, whose axes are proportional to the corresponding *velocity dispersions.* Such a figure with axes oriented along the radial direction toward or away from the Galactic centre, the transverse direction in the Galactic plane, and the normal direction perpendicular to the plane has traditionally been used to describe the velocity distribution of stars in the solar neighbourhood. Different kinds of stars sampling different *stellar populations* turn out to have somewhat different velocity ellipsoids. The mathematical expression for such a velocity distribution is:

$$f(u, v, w) \propto \exp\left[-\left(\frac{u^2}{2\sigma_u^2} + \frac{v^2}{2\sigma_v^2} + \frac{w^2}{2\sigma_w^2}\right)\right] \tag{52}$$

where u, v and w are the radial, transverse and normal components of the velocity vector of a star referred to the average motion of the whole population. The distribution is a combination of three independent Gaussians, whose standard deviations ("velocity dispersions") are denoted by σ_u, σ_v and σ_w, respectively.



As seen in the above diagrams, the real distributions are only approximately Gaussian, but the point to note is that the radial components – shown to the left – have a much larger dispersion than the normal components – shown to the right (the transverse components are shown in the middle). This is a clear indication that the third integral is isolating, at least as far as the orbits of local stars are concerned. The argument goes as follows. Assume that $\mathcal{E}$ and A_z are the only two isolating integrals. Then, according to the Jeans theorem, the distribution function is $\varphi = f(\mathcal{E}, A_z)$. If we take the x axis to be along the Sun's radial direction, and R_o is the distance to the Galactic centre, we can write

$$\varphi = f\left[\frac{1}{2}\left(u^2 + v^2 + w^2\right) + U(R_o, 0),\ R_o v\right] \tag{53}$$

for stars close to the Sun. It is easily seen that u and w appear in a completely symmetric way in Eq. (53), and therefore, in case we use an ellipsoidal distribution like in Eq. (52), we have to put $\sigma_u = \sigma_w$ so that the velocity ellipsoid has axial symmetry about the v axis. Thus the observed inequality of the axes indicates that a third integral must influence the actual velocity distribution.

Analytic expressions for this integral are known only for certain, special forms of the potential. We have already encountered the case of *spherical symmetry*, where

$$U(R, z) = f(R^2 + z^2)$$

and A_y and A_x are also integrals. Another such case is a *separable potential*, where

$$U(R, z) = f(R) + g(z)$$

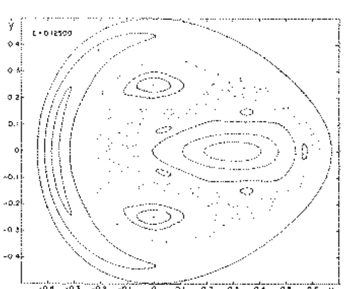
where f and g are arbitrary functions. We then have:

$$\ddot{z} = -\frac{\partial U}{\partial z} = -\frac{\partial g}{\partial z},$$

showing that the z motion can be integrated separately. The third integral is thus the energy of the z motion:

$$\mathcal{E}_z = g(z) + \frac{1}{2}w^2,$$

which is sometimes called the *Lindblad integral*. It reminds us of the case of plane-parallel geometry, treated in the preceding chapter. Thus, we see that there is a connection between the two assumptions such that both of them may be applicable, when we study motions near the central plane of a disk galaxy. We shall return to this connection below.



In a famous paper from 1964, Michel Hénon and Carl Heiles studied motions in another potential with a simple mathematical expression, chosen so as not to allow any trivial integral like in the cases just described. They instead performed numerical integrations and discovered that, for many values of the energy, the third integral would be both isolating and non-isolating, depending on which particular trajectory one considers. Looking at suitable projections of phase space, as in the plot to the left, one sees a mixture of 'integrable islands' with separate, closed curves, and an 'ergodic sea' where the orbit jumps around chaotically. This was one of the first demonstrations of *chaotic behaviour in dynamical systems* and is thus of great historical interest.

2.5.2 Planar, circular orbits

The aim of the rest of this chapter is to develop a theory for the motions of disk stars in the Galaxy – or similar stars in other spiral galaxies, of course. Thus we will consider a system, where the potential U has both axial symmetry and mirror symmetry around the central plane, *i.e.*, $U(R,-z) \equiv U(R,z)$. The orbits to be considered will be close to the plane of symmetry and nearly circular.

Let us start with circular orbits. From our assumption it follows that the vertical force component is zero everywhere in the central plane, so that if a star is situated in the plane with zero vertical velocity at a given moment, it will always stay in the plane. Moreover, thanks to the axial symmetry, the in-plane force component $-\partial U/\partial R\,(R,0)$ is always directed toward the origin, *i.e.*, the Galactic centre. We are thus back at the problem of planar motion under a central force, which we treated above for spherically symmetric systems. Hence, among the set of all possible orbits – in general 3-dimensional – there is a special class of planar, rosette-shaped orbits, and within this class there is a special family of circular orbits.

For any given radius R there is one such circular orbit, and we can compute the velocity V_c for this orbit by balancing the centrifugal and gravitational forces, which gives:

$$\boxed{V_c^2 = R\frac{\partial U}{\partial R}} \tag{54}$$

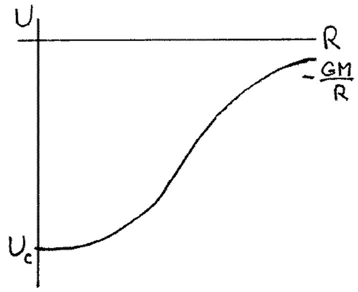
This is called the **circular velocity**. In terms of angular velocity, the corresponding quantity is $\omega_c = V_c/R$:

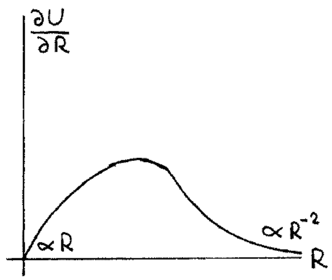
$$\boxed{\omega_c^2 = \frac{1}{R}\frac{\partial U}{\partial R}} \tag{55}$$

As far as observations of stellar motions in the solar neighbourhood are concerned, we can summarize the most important ones as follows.

- While different spectral types of stars have different average velocities with respect to the Sun (*i.e.*, the Sun has different *apex motions* relative to the different types), these velocities are almost always much smaller than our velocity relative to the Galactic centre.
- Therefore, we conclude that these stars (including the Sun) are moving with velocities close to the circular velocity, so their orbits can be treated as nearly circular in the above sense. The local, circular velocity defines what is often called the *dynamical Local Standard of Rest (LSR)*, while the average velocity of a given stellar type is the *kinematic LSR* of those stars.
- There is a trend for older stellar types to have larger velocity dispersions and to have mean velocities that lag behind the Galactic rotation. The latter phenomenon is called the *asymmetric drift*, and it is caused by the systematic deviation from circular orbits – in particular the ellipticities of the Galactic orbits.

Observations of stars outside the immediate solar neighbourhood reveal some features of the **rotation curve** of the Galaxy, *i.e.*, how V_c or ω_c vary as functions of R. Let us first show the theoretically expected appearance of this curve, based on the limiting cases of very small and very large orbits.





Close to the Galactic centre we can assume a potential close to that of spherical symmetry, so from the Poisson equation:

$$3\frac{\partial^2 U}{\partial r^2} \simeq 4\pi G\rho_c$$

and consequently:

$$U \simeq U_c + \frac{2}{3}\pi G\rho_c R^2$$

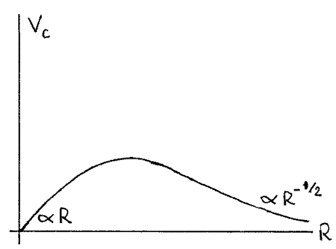
At large distances we have:

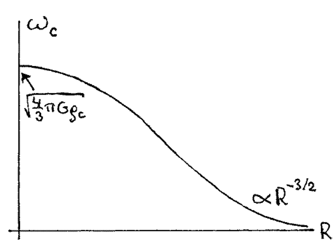
$$U \simeq -\frac{GM}{R}$$

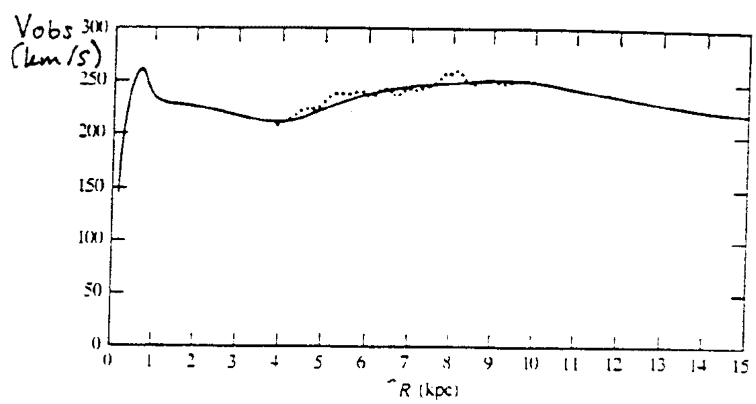
where M is the total mass of the Galaxy, which in this case acts like a point mass. From the above we realize the general shape of $U(R)$. We can also derive $\partial U/\partial R$, which has the asymptotic expressions:

$$\frac{4}{3}\pi G\rho_c R \qquad \text{and} \qquad +\frac{GM}{R^2}$$

From this we can derive $V_c(R)$ and $\omega_c(R)$, as shown below. Note that V_c passes through a maximum and eventually drops toward zero like $R^{-1/2}$, as appropriate for Keplerian orbits. However, comparison with the observed rotation curve of the Galaxy shows a poor fit. The observed curve exhibits a very flat and extended maximum and, as far out as the observations extend, does not approach the expected Keplerian decrease. The circular velocity stays at $\sim 250-300$ km/s in a very broad interval. There is a clear disagreement between this behaviour and the distribution of luminous matter (*i.e.*, mainly stars), which drops much faster in the outward direction. Such a disagreement is also seen in other disk galaxies like M31, and the preferred solution is to introduce a **dark halo** component into the potential functions. The large extent of these dark haloes allows the rotation curves to stay at a high level even at the outskirts of the visible stellar disks.





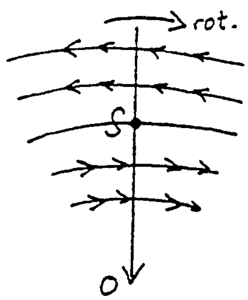


The circular angular velocity decreases outward in both the theoretical and observed cases. Thus the Galaxy does not at all rotate like a rigid body – the central parts rotate much faster than the exterior parts. This is a general phenomenon in many astrophysical contexts and is called **differential rotation**. We shall deal with it in the next Section. Let us now state the commonly accepted values for the dynamical LSR velocity, *i.e.*, the circular velocity of the solar neighbourhood. Recent measurements indicate:

$$\begin{aligned} R_o &\simeq 8 \text{ kpc} \\ V_o &\simeq 220 \text{ km/s} \end{aligned}$$

i.e., $\omega_o \simeq 27$ km/s/kpc, and the revolution period $P_o = 2\pi/\omega_o \simeq 240$ Myr.

2.5.3 Differential rotation and the Oort constants

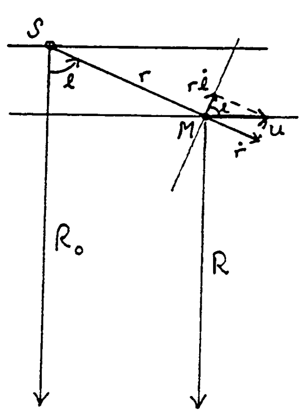


If we look at the Galaxy from the south pole so that it rotates clockwise, and if we place ourselves in a frame that rotates with the local circular angular velocity so that the dynamical LSR is at rest, the surrounding parts of the Galaxy will feature a shearing motion as indicated to the left. This is of course under the assumption of circular orbits. Outside the solar circle, the objects will drift counterclockwise due to their smaller angular velocity, and inside this circle the drift is clockwise.

From the observational point of view it is convenient to divide the motion of a star into the **radial velocity** (projection on the line of sight) and the **proper motion** (projection perpendicular to the line of sight). The methods of observing are of course very different in the two cases. Let us now calculate these two velocity components relative to the dynamical LSR for a star, which we assume to be moving in the Galactic plane on a circular orbit. The position of the star is given by its *distance* r and its longitude ℓ with respect to the Galactic centre, *i.e.*, the *Galactic longitude*. If the star is at a galactocentric distance R that is close to R_o, we have to first approximation:

$$\omega_c - \omega_o \simeq \left(\frac{d\omega_c}{dR}\right)_o \cdot (R - R_o) \tag{56}$$

yielding the apparent angular velocity around the origin. Multiplying by R, we get the apparent linear velocity:



$$u \simeq R \left(\frac{d\omega_c}{dR}\right)_o \cdot (R - R_o)$$

The components of this velocity are:

$$\begin{cases} \dot{r} &= u \sin\ell \\ r\dot{\ell} &= u \cos\ell \end{cases}$$

Since $R \simeq R_o$ and $R - R_o \simeq -r\cos\ell$, we get:

$$\begin{cases} \dot{r} &= -R_o \left(d\omega_c/dR\right)_o r \sin\ell \cos\ell \\ \dot{\ell} &= -R_o \left(d\omega_c/dR\right)_o \cos^2\ell \end{cases} \tag{57}$$

Here $\dot{r}$ corresponds to the observed radial velocity, but $\dot{\ell}$ does not correspond to the observed proper motion, because this angular velocity is defined in a rotating frame, while observed proper motions refer to a fixed frame. Since the $\dot{\ell}$ frame rotates with angular velocity $-\omega_o$, we would get the proper motion from: $\mu = \dot{\ell} - \omega_o$.

But, moreover, note that reality is not that simple. Observed velocities also include the Sun's motion relative to the dynamical LSR, and the corresponding residual velocities of the observed stars. One must therefore correct for these before applying the above formulae.

We now introduce:

$$\boxed{\begin{array}{rcl} A & = & -\frac{1}{2} R_o \left(d\omega_c / dR \right)_o \\ B & = & A - \omega_o \end{array}} \tag{58}$$

and Eqs. (57) thus transform into:

$$\boxed{\begin{array}{rcl} \dot{r} & = & A\, r \sin 2\ell \\ \mu & = & A \cos 2\ell + B \end{array}} \tag{59}$$

A and B are called the **Oort constants**. These formulae have been quite important for revealing the local part of the Galactic rotation curve. The method is to plot the corrected stellar velocities against Galactic longitude – in the case of $\dot{r}$ also dividing by the distance – and derive A and B from these plots. As seen from Eqs. (58), knowledge of R_o, A and B allows us to calculate ω_o and $(d\omega_c/dR)_o$.

By rearranging Eqs. (58), we get:

$$\boxed{\begin{array}{rcl} A - B & = & \omega_o \\ A + B & = & -\left(dV_c / dR \right)_o \end{array}} \tag{60}$$

i.e., expressed in words, the difference of the Oort constants gives the local, circular angular velocity of Galactic rotation, and the sum gives minus the local slope of the rotation curve. The values of A and B are not known very accurately, but indications are that the rotation curve is locally flat to a good approximation, and thus $A \simeq -B$. From the above value of $\omega_o = 27$ km/s/kpc, we then get

$$\begin{array}{rcl} A & \simeq & 13.5 \text{ km/s/kpc} \\ B & \simeq & -13.5 \text{ km/s/kpc} \end{array}$$

Let us finally see how the Oort constants may be used to express the Poisson equation in the solar neighbourhood. In cylindrical coordinates, this equation reads:

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial U}{\partial R} \right) + \frac{1}{R^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2} = 4\pi G \rho$$

Due to the axial symmetry, the θ term vanishes, and the radial term can be written:

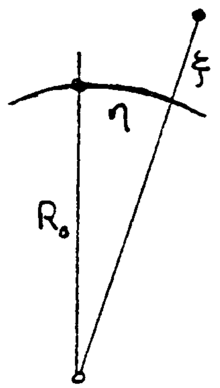
$$\frac{1}{R}\frac{dV_c^2}{dR} = 2\omega_c \frac{dV_c}{dR},$$

so in the solar neighbourhood we get:

$$\frac{\partial^2 U}{\partial z^2} - 2\left(A^2 - B^2 \right) = 4\pi G \rho \tag{61}$$

This shows that for a locally flat rotation curve, the Poisson equation holds in exactly the same form as we treated in the case of plane-parallel geometry. Any small deviation from this flat behaviour, if compatible with the observations, would require a small extra term in the Poisson equation, as given above with $A^2 - B^2 \neq 0$.

2.5.4 Epicyclic description of nearly circular orbits



For a perfectly circular orbit we would have:

$$\begin{cases} R &= R_o \\ \theta &= \omega_o t \\ z &= 0 \end{cases}$$

for an arbitrary value of R_o and $\omega_o = \omega_c(R_o)$. For the vicinity of such an orbit we can write:

$$\begin{cases} R &= R_o + \xi \\ \theta &= \omega_o t + \eta / R_o \\ z &= z \end{cases}$$

Note that η is a distance measured along a circle. The quantities ξ, η, z can be interpreted as coordinates in a reference frame that rotates with the angular velocity ω_o. We assume that all of them are very small in comparison to R_o, so that the particle moves on a nearly circular orbit close to the circle with radius R_o.

The equations of motion in cylindrical coordinates are:

$$\begin{cases} \ddot{R} - R\dot{\theta}^2 &= -\partial U / \partial R \\ 2\dot{R}\dot{\theta} + R\ddot{\theta} &= 0 \\ \ddot{z} &= -\partial U / \partial z \end{cases} \tag{62}$$

Substituting and developing the right-hand members in the neighbourhood of $(R_o, 0)$:

$$\begin{cases} \ddot{\xi} - (R_o + \xi)\left(\omega_o + \frac{\dot{\eta}}{R_o}\right)^2 &= -\frac{\partial U}{\partial R}(R_o, 0) - \frac{\partial^2 U}{\partial R^2}(R_o, 0)\,\xi - \frac{\partial^2 U}{\partial R \partial z}(R_o, 0)\,z - \dots \\ 2\dot{\xi}\left(\omega_o + \frac{\dot{\eta}}{R_o}\right) + (R_o + \xi)\frac{\ddot{\eta}}{R_o} &= 0 \\ \ddot{z} &= -\frac{\partial U}{\partial z}(R_o, 0) - \frac{\partial^2 U}{\partial z \partial R}(R_o, 0)\,\xi - \frac{\partial^2 U}{\partial z^2}(R_o, 0)\,z - \dots \end{cases}$$

We linearize, thus neglecting all terms of second or higher order. Also, due to the mirror symmetry about the $z = 0$ plane, we have:

$$\frac{\partial U}{\partial z}(R_o, 0) = \frac{\partial^2 U}{\partial R \partial z}(R_o, 0) = \frac{\partial^2 U}{\partial z \partial R}(R_o, 0) = 0$$

Since we also have:

$$R_o \omega_o^2 = \frac{\partial U}{\partial R}(R_o, 0),$$

we finally get:

$$\begin{cases} \ddot{\xi} - 2\omega_o\dot{\eta} - \omega_o^2\xi &= -\left(\partial^2 U / \partial R^2\right)_o \cdot \xi \\ \ddot{\eta} + 2\omega_o\dot{\xi} &= 0 \\ \ddot{z} &= -\left(\partial^2 U / \partial z^2\right)_o \cdot z \end{cases} \tag{63}$$

The second derivatives of U are taken in a given point and are thus to be regarded as constants. We have a homogeneous system of linear differential equations with constant coefficients for ξ, η, z. Note that the above equations degenerate into those for a plane-parallel system in case U depends only on z.

The system is *separable* – the first two equations only contain ξ and η, and the third one only z. The latter can be integrated directly, yielding:

$$\boxed{z = z_o \cos \omega_z \left(t - t_2\right)} \tag{64}$$

where z_o and t_2 are integration constants, and $\omega_z^2 = \left(\partial^2 U / \partial z^2\right)_o$. This is a sinusoidal z motion caused by a restoring force toward the Galactic plane that is proportional to the distance from the plane. This behaviour is predicted, for instance, by the isothermal model discussed above.

The second equation can also be integrated:

$$\dot{\eta} + 2\omega_o \xi = a \qquad (a = \text{const.})$$

and the first one then becomes:

$$\ddot{\xi} = -\left[\left(\partial^2 U / \partial R^2\right)_o + 3\omega_o^2\right] \xi + 2\omega_o a$$

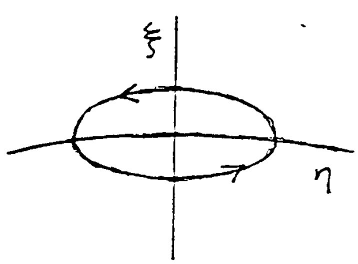
Let us put:

$$\kappa_o^2 = \left(\partial^2 U / \partial R^2\right)_o + 3\omega_o^2, \tag{65}$$

which is a constant. The solution of the last differential equation is hence an oscillation with frequency κ_o, and we get the general solution for ξ and η as:

$$\boxed{\begin{aligned} \xi &= \tfrac{2\omega_o a}{\kappa_o^2} + c\, \cos \kappa_o \left(t - t_o\right) \\ \eta &= a \left(1 - \tfrac{4\omega_o^2}{\kappa_o^2}\right) \left(t - t_1\right) - \tfrac{2\omega_o c}{\kappa_o}\, \sin \kappa_o \left(t - t_o\right) \end{aligned}} \tag{66}$$

where a, c, t_o and t_1 are integration constants.



Let us first assume that $a = 0$. The remaining terms are proportional to c and yield a quasi-elliptic motion in the (ξ, η) plane. The axial ratio is $2\omega_o / \kappa_o$, and the sense of motion is opposite to the orbital motion around the Galactic centre. The angular velocity is κ_o, and this is in general different from ω_o. The curve described in the (ξ, η) plane would be an ellipse if it were not for the curvature of the η axis. It is called an **epicycle** in analogy with the ancient description of planetary orbits in terms of circles and epicycles, and κ_o is referred to as the **epicyclic frequency**.

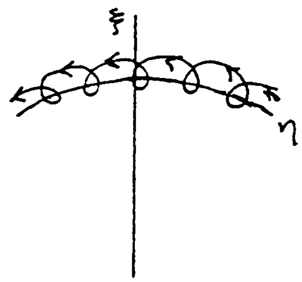
We have:

$$\left(\partial U / \partial R\right)_o = R_o \omega_o^2 \quad ; \quad \left(\partial^2 U / \partial R^2\right)_o = \omega_o^2 + 2 R_o \omega_o \left(\partial \omega / \partial R\right)_o$$

and thus:

$$\kappa_o^2 = 4\omega_o^2 - 4\omega_o A = -4\omega_o B, \tag{67}$$

yielding the relation between κ_o and the Oort constants. We see that the epicyclic frequency is determined by the local properties of the Galactic rotation curve. Using the above estimates, we get $\kappa_o \simeq 38$ km/s/kpc. The axial ratio of the local epicycle is thus $54/38 \simeq 1.4$.



If we instead assume $c = 0$ and let a take a non-zero value, we get a constant value of ξ and a uniformly changing value of η. This is a **circular drift motion** around the Galactic centre. The reason is simply the differential rotation of the Galaxy. If we choose $a \neq 0$, we place the orbit at an average distance $R \neq R_o$, and thus we get a drift in Galactic longitude. In general we have to consider both a and c as non-zero, and the drift and epicyclic motions have to be combined. This is illustrated in the figure to the left.

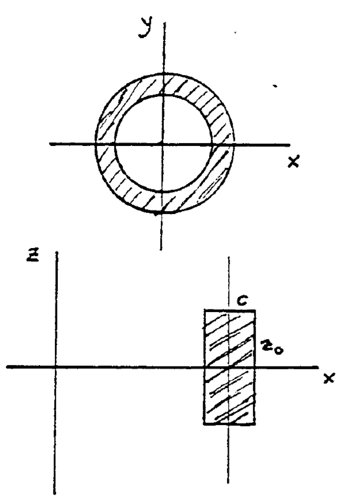
Note that this appearance of the motion is due to our use of a rotating frame of reference. In a fixed frame we would not find an epicycle but a rosette curve. Our ξ coordinate oscillates between

$$\frac{2\omega_o a}{\kappa_o^2} \pm c$$

and if $a > 0$, ξ will on the average be positive, *i.e.*, the orbit is on the average situated outside R_o – the motion around the Galactic centre is thus slower, yielding the apparent retrograde drift seen in the figure.

We can summarize the general motion, now seen in a fixed frame, as a combination of three periodic motions:

1. a circular rotation around the Galactic centre;
2. a small-scale quasi-elliptic motion in the Galactic plane, opposite to the direction of rotation;
3. an oscillation perpendicular to the plane.



The angular velocities are, respectively:

$$\begin{aligned}
\omega_c^2 &= \frac{1}{R}\frac{\partial U}{\partial R} \\
\kappa^2 &= \frac{\partial^2 U}{\partial R^2} + \frac{3}{R}\frac{\partial U}{\partial R} \\
\omega_z^2 &= \frac{\partial^2 U}{\partial z^2}
\end{aligned}$$

The three frequencies feature different derivatives of U and are thus mutually independent and may have any values. In general those values are not commensurable. Therefore, for instance, after an even number of rotations around the Galactic centre, the two other motions have not completed an even number of periods. The projection of the $3D$ orbit onto the (x, y) plane will fill up an annulus in the way of a general rosette orbit, and the orbit itself will fill up a *toroidal volume* with a rectangular cross-section.

The fact that the orbit in general fills up a volume implies that there are two non-isolating integrals. But we can also find three isolating integrals in the shape of integration constants:

$$
\begin{aligned}
a &= 2\omega_o\xi + \dot{\eta} \\
c^2 &= \left[\left(1 - \frac{4\omega_o^2}{\kappa_o^2}\right)\xi - \frac{2\omega_o}{\kappa_o^2}\dot{\eta}\right]^2 + \frac{\dot{\xi}^2}{\kappa_o^2} \\
z_o^2 &= z^2 + \frac{\dot{z}^2}{\omega_z^2}
\end{aligned}
$$

Thus we arrive at an important conclusion: **In the vicinity of circular Galactic orbits the third integral is isolating.** Returning to the velocity ellipsoid with its three different axes, we see that this can be explained, as long as the above, linearized theory os nearly circular motions is valid.

Instead of a we can use:

$$\overline{R} = R_o + \frac{2\omega_o a}{\kappa_o^2}$$

This is simply the average value of R independent of which value we use for R_o. The three integrals thus have a simple physical interpretation, namely, the average radius, the amplitude of the radial oscillations, and the amplitude of the vertical oscillations. The distribution function can be written:

$$\varphi = f\left(\overline{R}, c^2, z_o^2\right)$$

To be continued