Inverse problems

Intro
Lecture 1

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Introduction

- Typical situation where an inverse problem approach comes naturally is remote sensing.
- Remote sensing differs from laboratory science: *one cannot dissect and specially prepare the study object*.
- Astrophysics has a lot in common with remote sensing.

Plan

During this course we will study:

- Specific applications from various areas that lead to inverse problems
- Describing inverse problems with integral equations
- Solving integral equations
- Learning the concept of regularization
- Applying inverse problem approach to astrophysics

Plan (cont'd)

We will use a few astrophysical and non-astrophysical examples to learn:

- How to formulate a problem as in integral equation and to solve it
- How to choose an appropriate functional form and value of regularization
- How to select an approriate numerical method for solving an inverse problem

Plan (cont'd)

Literature:

Selected papers from e-journal "Inverse Problems", ApJ and A&A

Plan (cont'd)

Students will have to:

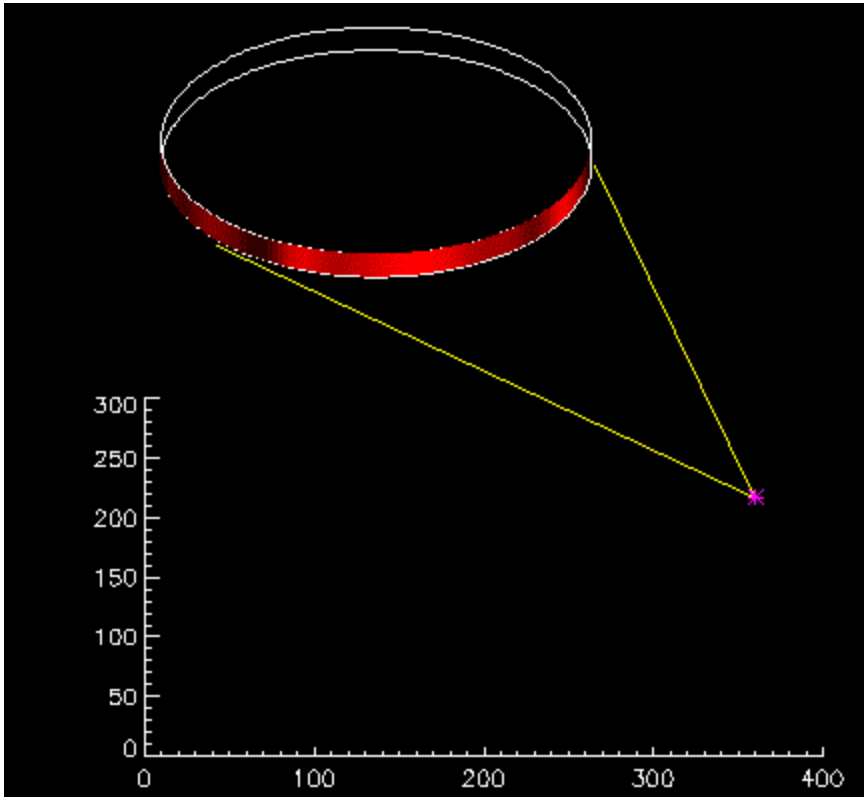
- Do the home work consisting of making (small) computer codes reproducing examples described in class
- Select one paper from the proposed list, study it and present in the class
- Formulate a project based on your own area of research and suggest how to solve it using inverse problem approach
- Present the results in the class
- **No exam!**

Papers

- ***Difficulties with Recovering the Masses of Supermassive Black Holes from Stellar Kinematical Data***
  Monica Valluri, David Merritt and Eric Emsellem, 2004, ApJ **602**, 66
- ***Wavelet methods in multi-conjugate adaprive optics***
  T Helin and M Yudytskiy, 2013, IP **29**, 085003
- **Convergence rates in expectation for Tikhonov-type regularization of inverse problems with Poisson data**
  Frank Werner and Thorsten Hohage, 2012, IP **28**, 104004
- ***Inversion of second-difference operators with application to infrared astronomy***
  M Bertero, P Boccacci and M Robberto, 2003, IP **19**, 1427
- ***Changes in the Subsurface Stratification of the Sun with the 11-Year Activity Cycle***
  S. Lefebvre and A. G. Kosovichev, 2005, ApJ **633**, L149***Dust Grain Size Distributions from MRN to MEM***
  Geoffrey C. Clayton, Michael J. Wolff, Ulysses J. Sofia, K. D. Gordon and K. A. Misselt, 2003, ApJ **588** 871
- **Doppler Imaging of stellar magnetic fields. I. Techniques**
  Piskunov, N.; Kochukhov, O. 2002, A&A **381**, 736
- **Doppler Imaging of stellar magnetic fields. II. Numerical experiments**
  Piskunov, N.; Kochukhov, O. 2002, A&A **388**, 868

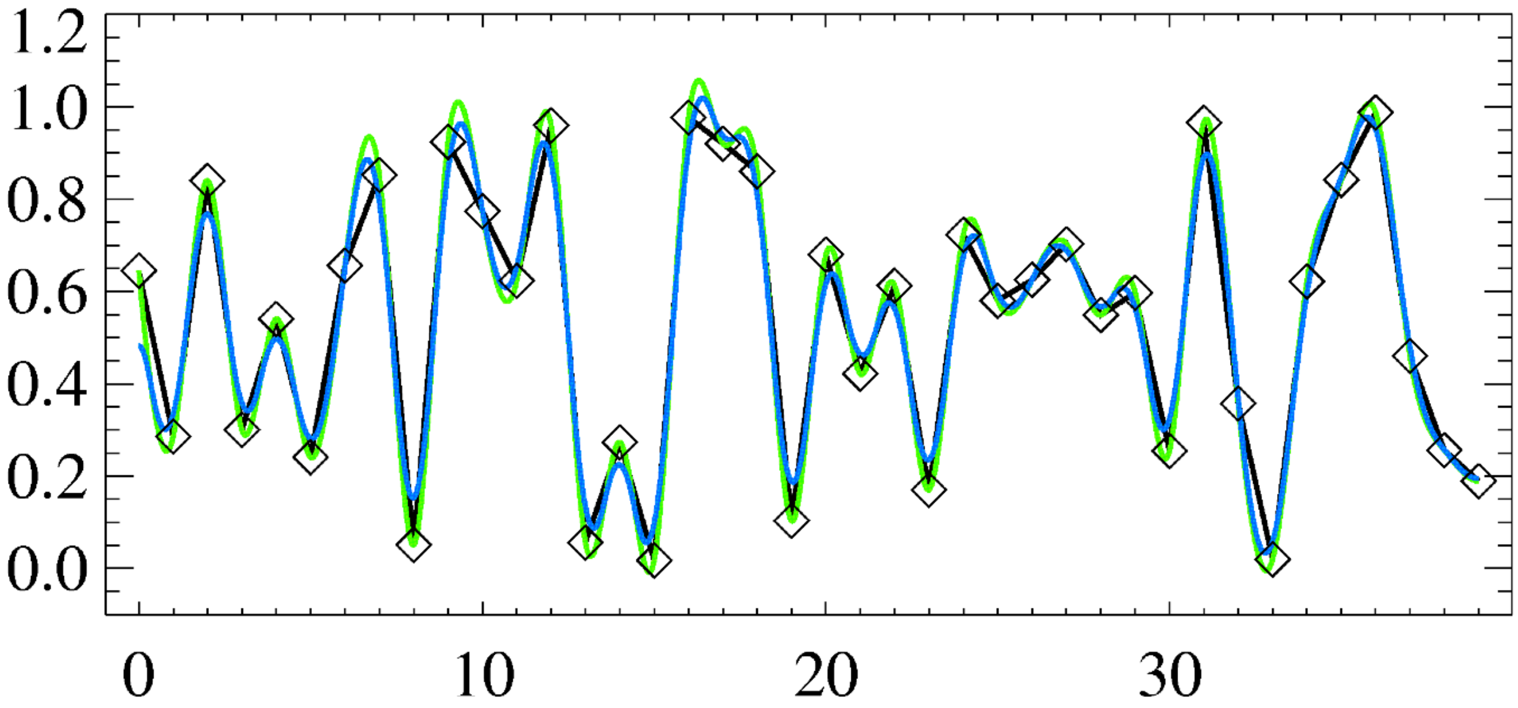
Remote sensing

- Observations and object data are in different spaces. Example: for a rotating ring the data is in spatial coordinates but the observations are in time domain.
- Try to write the equation connecting observations to the brightness of the ring



Mathematical functions

Interpolation between noisy data points



Some math

Remote sensing problem as an integral equation (some examples):

1. Radiative transfer in semi-infinite non-absorbing medium:

$$I^{+}(x) = \int_{0}^{x} s(l)dl$$

2. Image smoothing with a box-like kernel:

$$R_{ij} = \iint I(x, y) \prod_{ij} dxdy$$

In more general case:

$$g(x) = af(x) + c \int_{A(x)}^{B(x)} K(x, z) f(z) dz$$

What about solutions?

1. RT is trivial: $s(x) = \dfrac{dI^+}{dx}$
2. Pixel deconvolution is degenerate.
   If I is a solution then for any periodic in x and y function L such that:

   $L(x, y) = L(x + n \cdot \Delta, y + m \cdot \Delta)$ for any integer n, m $\quad I + L$ is also a solution.
   Try to test this.

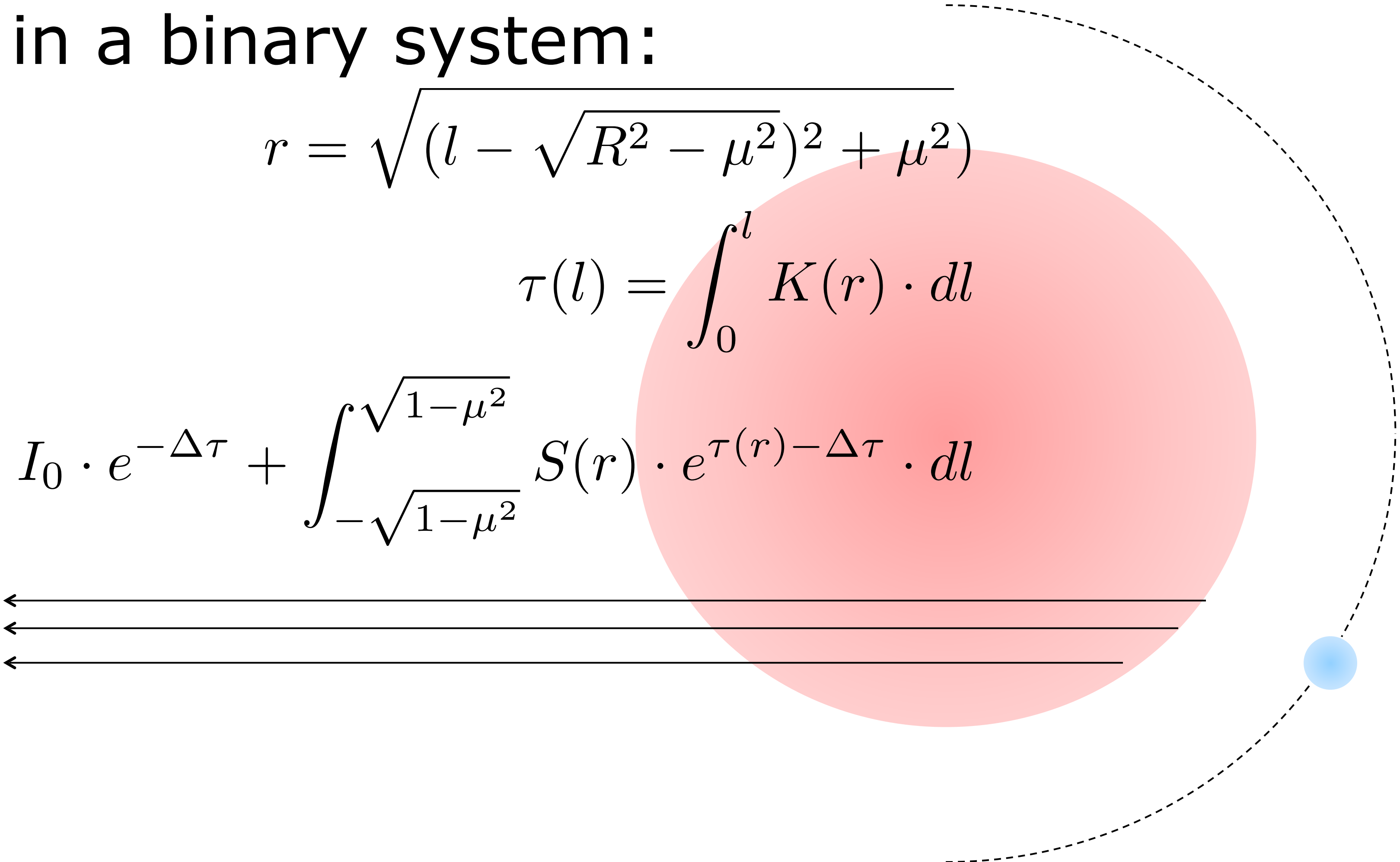
More examples

Atmospheric structure of a giant star in a binary system:

$$r = \sqrt{(l - \sqrt{R^2 - \mu^2})^2 + \mu^2}$$

$$\tau(l) = \int_0^l K(r) \cdot dl$$

$$I(\mu) = I_0 \cdot e^{-\Delta\tau} + \int_{-\sqrt{1-\mu^2}}^{\sqrt{1-\mu^2}} S(r) \cdot e^{\tau(r)-\Delta\tau} \cdot dl$$



Functionals

- "Forward problem". The equations above can be written in operator form: $\mathbf{K}f = g$ which allows evaluating g for a given f. This procedure is called **forward problem**.
- Linear operators. All examples above belong to a linear type of integral equations. Properties of linear operators:

$$\mathbf{K}(\alpha \cdot p + \beta \cdot q) = \alpha \cdot \mathbf{K}p + \beta \cdot \mathbf{K}q = \alpha \cdot u + \beta \cdot v$$

- Think of other remote sensing measurements and linear operators that describe them.

Classification of linear integral equations

- Fredholm equation:

$$g(x) = af(x) + c\int_A^B K(x,z)f(z)dz$$

Difference

- Volterra equation:

$$g(x) = af(x) + c\int_A^{B(x)} K(x,z)f(z)dz$$

- If $a=0$ then both equations are referred to as *first kind integral equations*.

Non-linear functionals

- Often, the problem is non-linear:

$$g(x) = af(x) + c\int_{A(x)}^{B(x)} K(x, z, f(z))dz$$

- Can you come up with examples?
- "Inverse problem." We are interested in the function *f*. In principle we may hope to construct an *inverse operator*:

$$f = \mathbf{K}^{-1}g$$

Sometimes one can easily find an inverse operator. A simple radiative transfer problem above is a good example. In many cases that is not possible. In particular, Fredholm problems of the first kind do not allow any simple way of constructing an inverse operator.