Lecture 2

History and the concept of an ill-posed problem

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2014

Ill-posed problems

- Jacques Hadamard introduced the concept of an ill-posed problem in 1902.
- Ill-posed problem means that there are multiple f that fulfil the operator equation

$$\mathbf{K}f = g$$

Les questions mal posées

According to Hadamard a well-posed problem:

- Has a solution
- The solution is unique
- The solution is continuous in respect to the parameters of the problem

Example

Heat conduction problem

Heat equation is a parabolic PDE:

$$\frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

Initial conditions:

$$u(0, x) = f(x)$$

Boundary conditions:

$$u(t, 0) = 0$$

Solving heat equation (1D)

First proposed by Joseph Fourier (1822)

Let's look for solution in the form:

$$u(t,x) = X(x)T(t)$$

then the equation can be re-written as:

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}$$

and split into two equations:

$$T'(t) = -\lambda k T(t)$$ and $$X''(x) = -\lambda X(x)$$

Solving heat equation (cnt'd)

The solution for a finite domain has a form:

$$T(t) = Ae^{-\lambda k t}$$
$$X(x) = B\sin(\sqrt{\lambda}x)$$

where λ can take discrete positive values.

This was the beginning of the Fourier analysis!

Hadamard was driven by the puzzle that the heat equation cannot be solved uniquely backward in time!

Solving an inverse problem

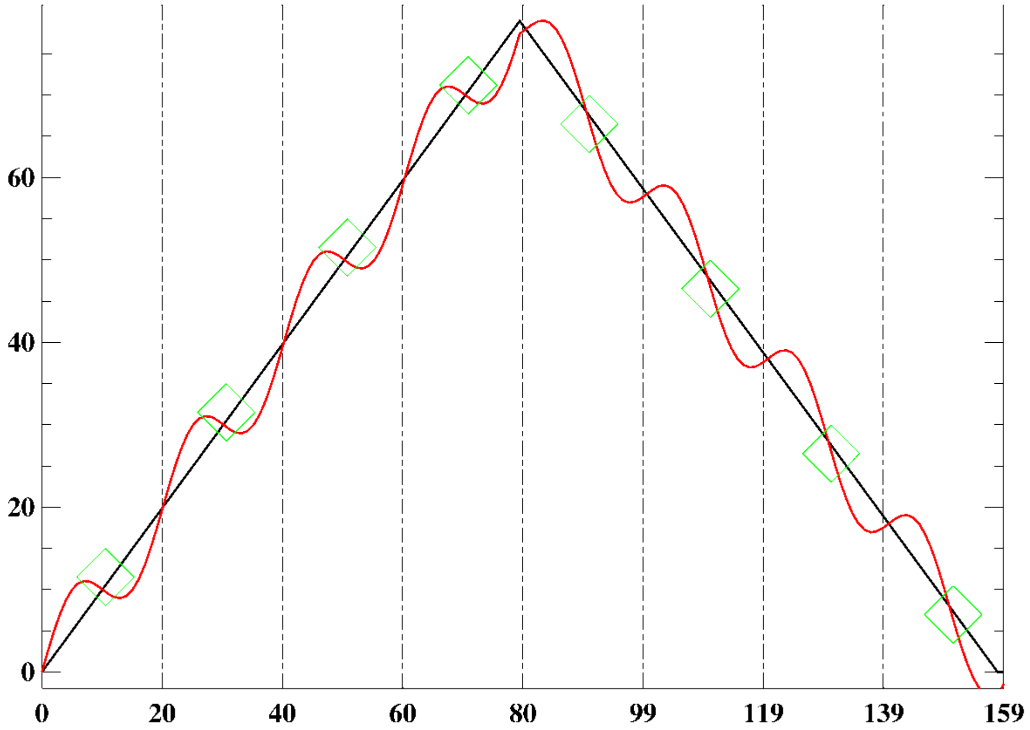
- The first example of RT in non-absorbing medium:

$$I(x) = \int_0^x S(x')dx'; \quad S(x) = \frac{dI}{dx}$$

- The second example (fixed pixel sampling) is not that easy:

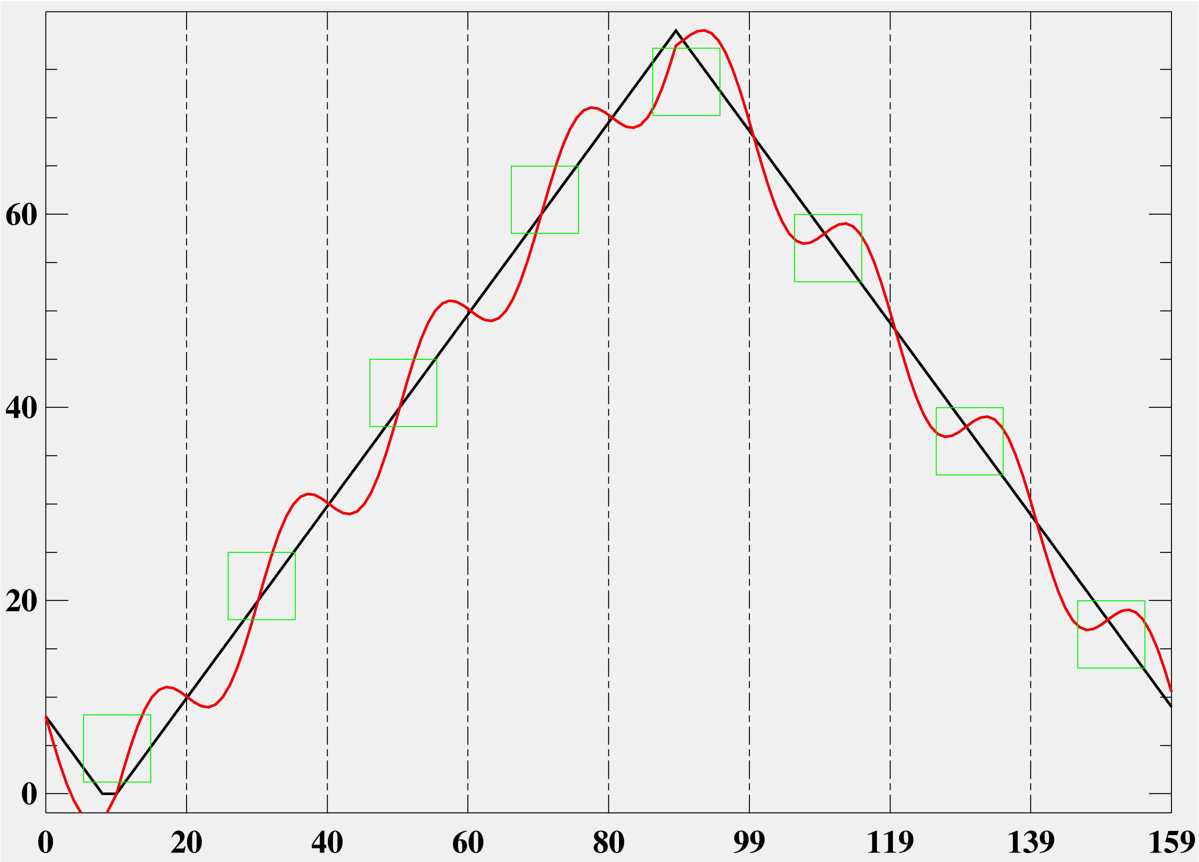
$$R_i = \int_{x_i-\Delta/2}^{x_i+\Delta/2} I(x)dx, \quad i = 1, \ldots, N$$

Clearly, if we have a single realization of *R*'s we will not be able to distinguish the following functions:

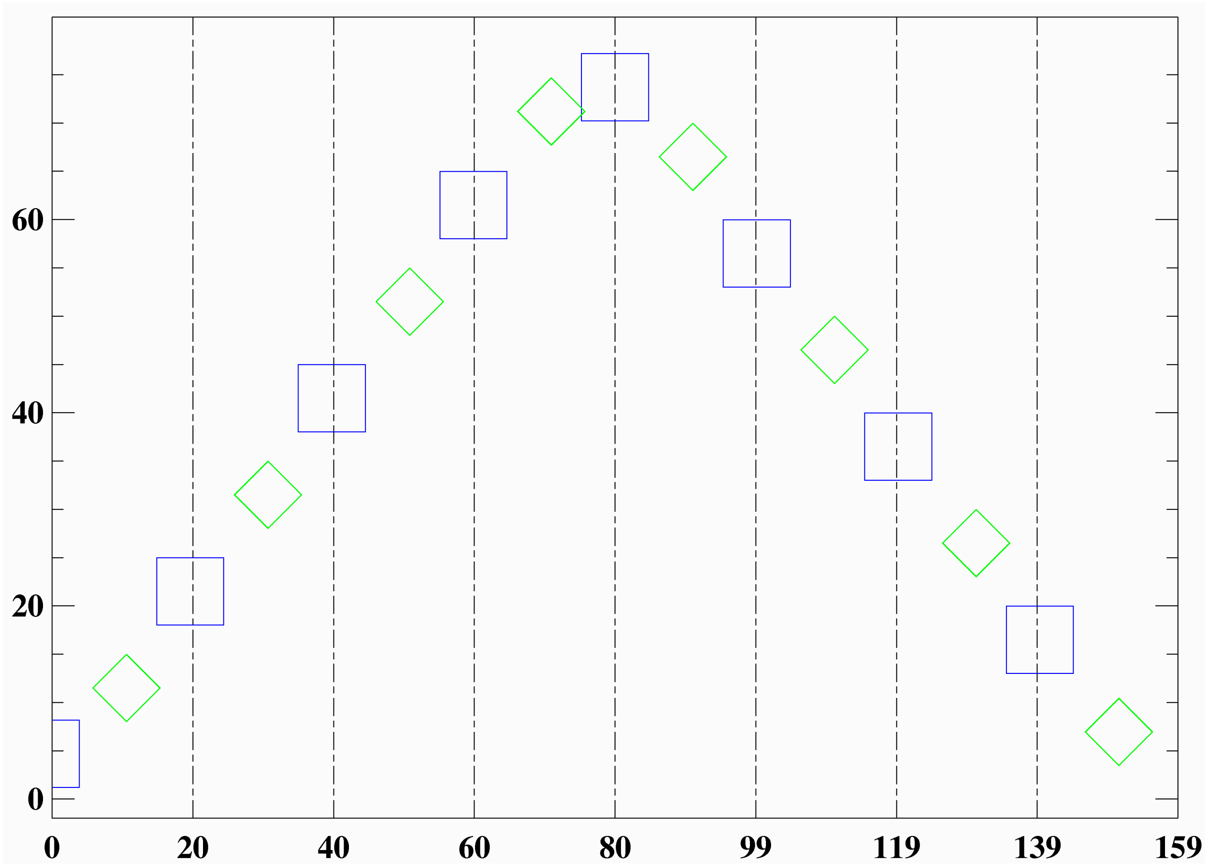


although the *R*'s for them are very close.

Even if we try several realizations of *R*'s with sub-pixel offset:



Measurements change but we still see no difference between the two solutions:



So far we have "guessed" *f* and computed *g* - forward problem. How can we solve an inverse problem? We can use optimization approach:

$$\Phi(f) \equiv \left\| \mathrm{K} f - g \right\| = \min$$

The norm can be, for example, a least square discrepancy measure:

$$\Phi(f) \equiv \int \omega(x) \left[\mathrm{K} f(x) - g(x) \right]^2 dx$$

The search for minimum of Φ can be performed in a number of ways. The simplest way is the gradient search. Numerically that means a search on a grid of x_i with $f_i \equiv f(x_i)$:

$$f_i^{n+1} = f_i^n - \Delta^n \frac{\partial \Phi(f^n)}{\partial f_i^n}$$

provided that the partial derivatives of Φ exist. The upper index *n* here counts iterations and Δ^n is the step size. The gradient method as just a 1D search along the steepest slope.

We can easily evaluate the gradient vector:

$$\Phi(f) \equiv \sum_o \sum_i \left[\sum_{i-\delta<x+o<i+\delta} f_{x+o} - g_i \right]^2$$

$$\frac{\partial \Phi}{\partial f_x} = 2 \sum_o \left[\sum_{k-\delta<x+o<k+\delta} f_{x+o} - g_k \right], \quad \text{for } |k - x - o| < \delta$$

for our problem but we have no way to determine *a priori* the step size Δ^n. Instead on each iteration we have to search for the optimal step.

Let's see how it works:

1° Select the grid for sampling *x*

2° Select the initial guess for *f*

3° Evaluate $\nabla\Phi$

4° Search for Δ^n that results in minimum of Φ

5° Repeat from 3 until convergence is achieved.

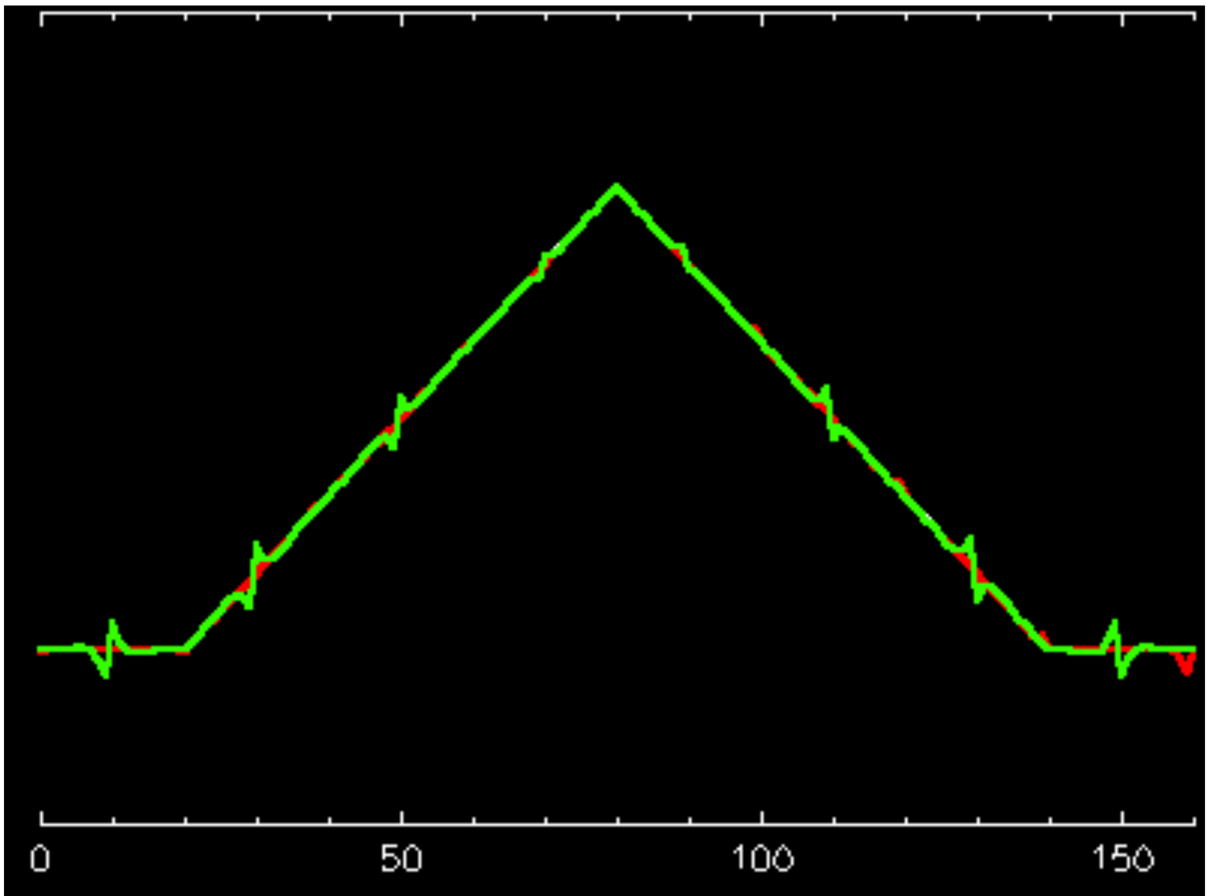
This will work for a single as well as for multiple sets of *g*'s e.g. as in sub-pixel offset.

The question is if the solution is unique and stable in respect to the right hand side.

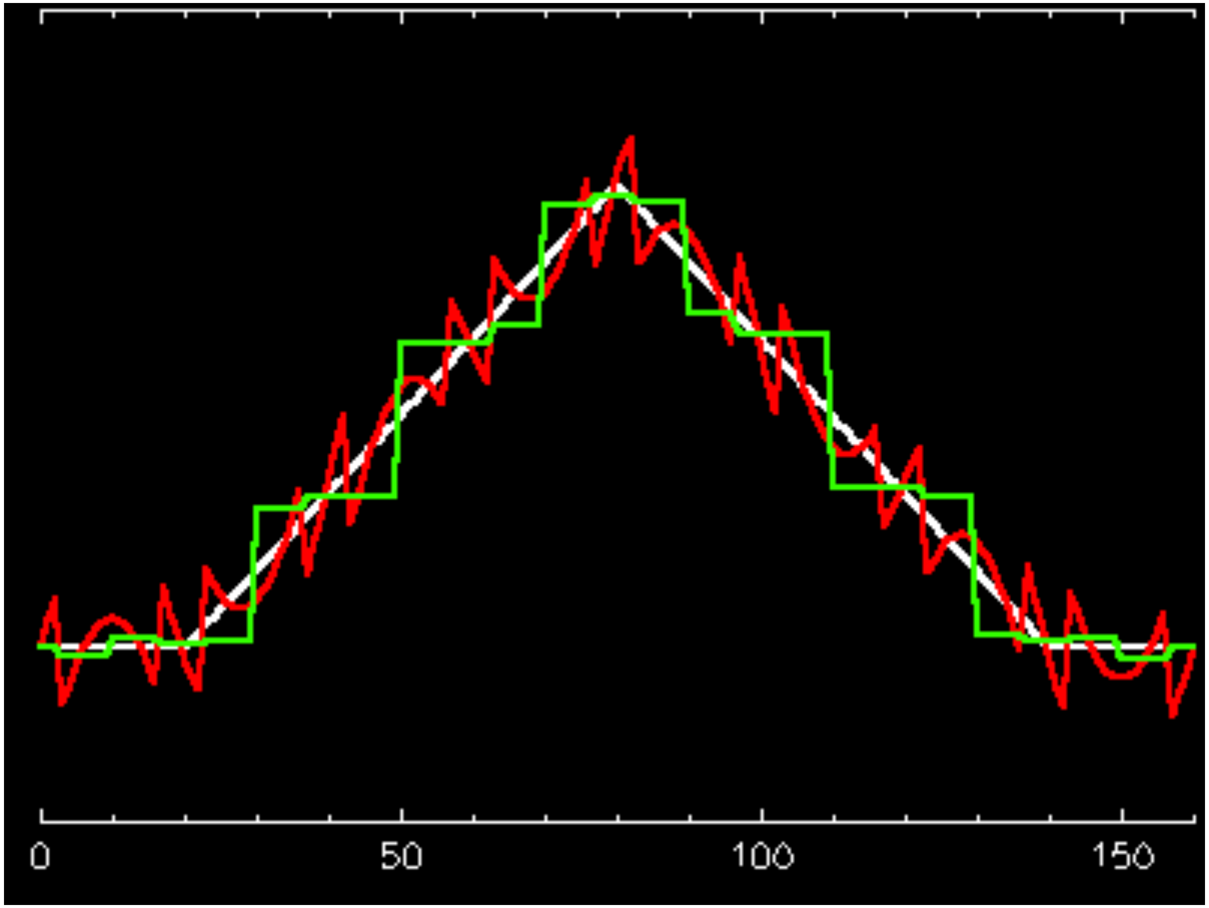
Let us get more specific:

The test function has triangular shape and is set on 160 grid points. It is sampled with 20 point wide pixels. We have a total of 7 measurements (*g*) for a given starting point. In the first attempt we use *g*’s corresponding to two offsets (3 and 17) giving the total of 14 values. A complete set of *g*’s include all possible offsets (0-19) resulting in 140 measurements.

One would intuitively think that in the first case the solution will be worse than a complete data set is used but if I start with a smooth initial guess I get this:



In the second case I started with a smooth line + a sin curve. The solution is much worse even though I have 140 data points instead of 14:



Conclusions:

- A minimization procedure allows to get a solution even when we don’t know K^{-1}.
- A simple gradient method works but slowly (why?). Find out about conjugate gradients techniques.
- The solution is dependent on the initial guess but two very different *f* ’s may give very similar *g*’s.

Astrophysical problems leading to Fredholm equation

- Instrumental profile or Point Spread function (convolution):

$$g(x) = \int_{-\infty}^{\infty} p(x-y) f(y) dy$$

In practice the instrumental profile is significantly different from zero in some limited range of x :

$$g(x) = \int_{-\Delta}^{+\Delta} p(x') f(x-x') dx'$$

Home work

- Read about [inverse heat conduction problem]
  and try understanding the problem
- Program a gradient search minimization
- Repeat class examples
- Answer questions:
  - Why do we get multiple solutions?
  - How many iterations does it take?
  - How do you stop?