Optimal filtering

1 Fitting data

We want to fit equispaced data points imposing Tikhonov regularization on the fitting function. We do this by minimizing a functional Ω:

$$\Omega \equiv \sum_i (f_i - g_i)^2 + \Lambda \cdot \sum_i (f_i - f_{i-1})^2 = \text{minimum}. \tag{1}$$

The solution of Eq. 1 can be found by setting all components of the gradient vector to 0:

$$\frac{1}{2}\frac{\partial \Omega}{\partial f_k} = (f_k - g_k) + \Lambda \cdot (2f_k - f_{k-1} - f_{k+1}) = 0. \tag{2}$$

We can re-write this system of linear equations in the form:

$$\sum_i A_{ij} f_i = g_j, \tag{3}$$

where:

$$A_{ij} = \begin{pmatrix} 1+\Lambda & -\Lambda & 0 & \dots & & 0 \\ -\Lambda & 1+2\Lambda & -\Lambda & & & \\ 0 & -\Lambda & 1+2\Lambda & -\Lambda & & \\ \dots & & & & & \\ & & & -\Lambda & 1+2\Lambda & -\Lambda \\ & & & & -\Lambda & 1+\Lambda \end{pmatrix} \tag{4}$$

Matrix A_{ij} is tri-digonal and, therefore, one can quickly solve Eq. 3 and find f_i.
You may try to program this and fit different data: sine curve, Gaussian, noise with different values of Λ.

2 Interpolation

Interpolation here means that we want to find f_l on an equispaced grid *which is denser than the grid of the original data.*
The Eq. 1 has to be re-written as following:

$$\Omega \equiv \sum_{i=0}^{N_{\text{data}}-1} (f_{i \cdot m} - g_i)^2 + \Lambda \cdot \sum_{l=0}^{N_{\text{fit}}-1} (f_l - f_{l-1})^2 = \text{minimum}, \tag{5}$$

where $m = N_{\text{fit}}/N_{\text{data}}$.

1. Follow the derivation for the case of fitting and find the equation of f_l in case of interpolation.
2. Write a program to find f_l for a given data, Λ and m
3. Construct data that samples a step-function and try to interpolate it with cubic spline and with the optimal filtering technique.