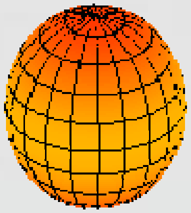
UPPSALA UNIVERSITET

Lecture 2:
Theoretical stellar models
1D static models

NORDITA Master Class 2006

Models of stars



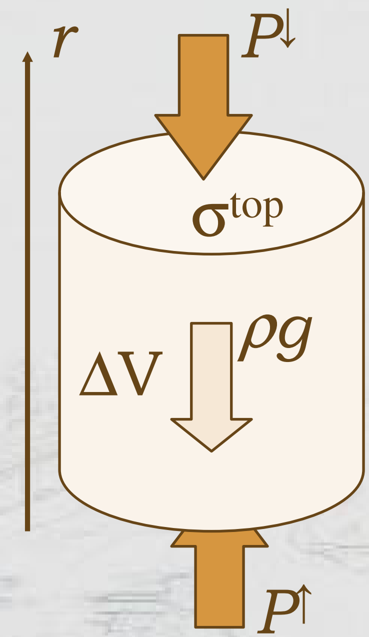
- First analytical model of a star: blob of gas described by hydrostatic equilibrium and simplified equation of state. These led to realize physical limits on stellar parameters (mass, temperature, rotation).
- Models of star formation and evolution including core nuclear reactions and gradual change of chemical composition. Allow e.g. assigning accurate age to a group of stars with the same origin.
- Models of stellar atmospheres including energy transport by radiation and convection. This are crucial for determination of physical parameters (temperature, mass, composition, rotation) of individual objects.

Under pressure

For a small volume element in the atmosphere the total force is:

$$F = -\rho g \Delta V - (P_{\text{gas}}^{\downarrow} + P_{\text{rad}}^{\downarrow}) \cdot \sigma^{\text{top}} +$$

$$+(P_{\text{gas}}^{\uparrow} + P_{\text{rad}}^{\uparrow}) \cdot \sigma^{\text{bottom}}$$

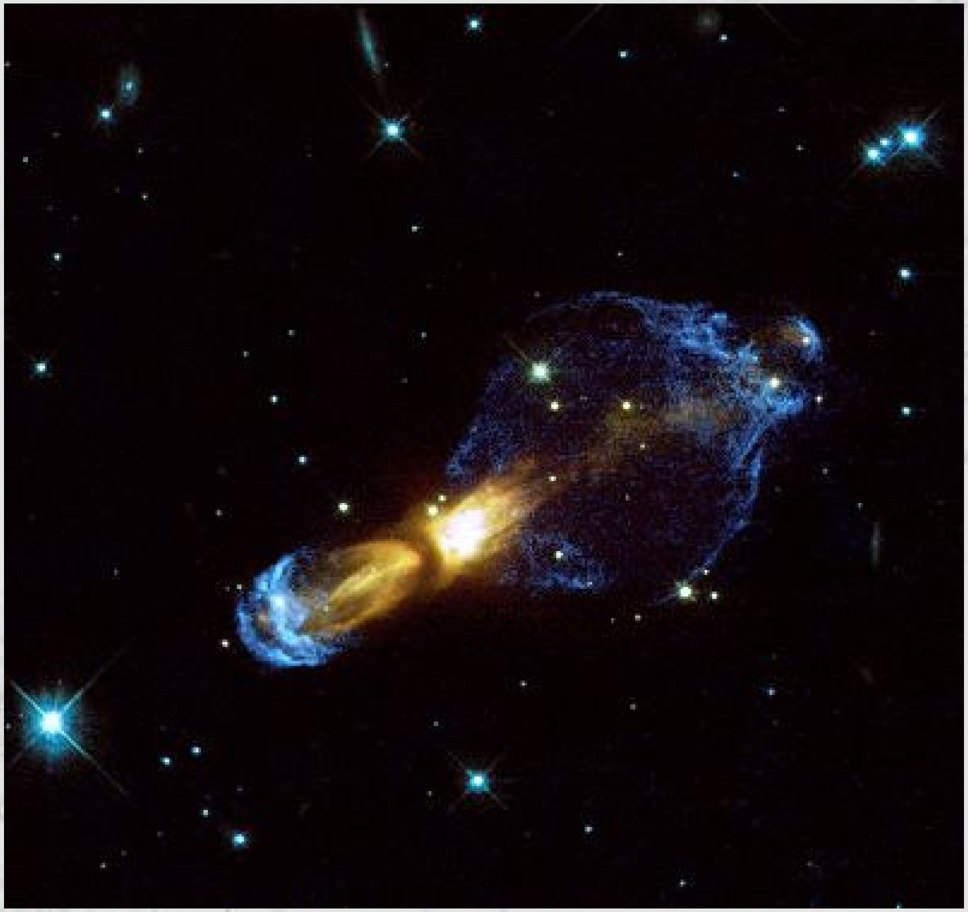


Small means that we can assume T, P_g, ρ and k_ν to be constant but large as compared to particles mean free path. In hydrostatic case the net force is zero: $F = 0$

The radiative pressure is significant in hot stars where the majority of the photons carries high momentum or …

… in stars with very low surface gravity

(Rotten Egg Nebula)



Simplest model

- In fact if we postulate a simple equation of state like e.g. polytrope $P = K\rho^{1+1/n}$ we get a closed system of equations:

$$\frac{d\left(P_{\text{gas}} + P_{\text{rad}}\right)}{dr} = -g(r)\rho$$

$$g(r) = \frac{G}{r^2}\int_0^r \rho 4\pi s^2 ds$$

$$P_{\text{gas}} = K\rho^{1+1/n}$$

- Assuming radiative pressure to be a constant fraction of a gas pressure we get the Eddington equation with n=3 which can be directly integrated assuming central stellar density.
- Try to combine all these equations in one assuming that

$$P_{\text{gas}} + P_{\text{rad}} \equiv (1+\beta)\cdot P_{\text{gas}}$$

Simplest model

- Eddington model gives dependence between mass and radius for stars given the central density
- This led to a major breakthrough in understanding why stars are not randomly distributed in mass-radius and mass-luminosity planes
- Still, these models are too primitive because we ignore the role of radiation, have unrealistic equation of state etc.

1D radiative models: Basic idea

Let us consider a relatively small volume element in the thin surface layer from which radiation can escape in reasonably short time. This layer is called *stellar atmosphere*. It is affected by the gravity, gas pressure and radiative pressure. Gas internal energy is characterized by local temperature. Cooling and heating are performed by coming and leaving radiation.

Example of the Solar atmosphere

- Solar radius $R_\odot \approx 7 \cdot 10^5$ km. Surface layer from which EM radiation can escape is $\approx 10^3$ km thick. Solar mass $M_\odot \approx 2 \cdot 10^{33}$ g $\Rightarrow$ acceleration of gravity (surface gravity g) = $2.7 \cdot 10^4$ cm/s^2 and it changes by less than 0.3% within the atmosphere. Surface gravity is often expressed in log cgs units.
  $\log g_\odot = 4.44$
- Temperature changes significantly (>2 dex) throughout the atmosphere.
  Effective temperature $\sigma T_{eff}^4 =$ ***total flux***
- Since temperature changes, pressure, density and composition varies as well!

Typical assumptions

1) Steady state
2) Constant gravity
3) Hydrostatic equilibrium
4) Most of the energy is transported by radiation
5) Local thermodynamic equilibrium
6) Plane-parallel or spherically symmetric geometry
7) No explicit convection
8) Homogeneous chemical composition
9) No wind, no accretion, no magnetic fields
10) No surface structures (spots of any kind)
11) No magnetic fields

Interaction between radiation and gas

- More realistic equation of state: Stellar material behaves like perfect gas: $P = kT \cdot N$ (k is Boltzmann constant). This holds for all particles in a unit volume and for each species separately, e.g. for electrons $P_e = kT \cdot N_e$
- The velocity distribution is Maxwellian: $N_{\text{part}}(\upsilon \to \upsilon + \Delta\upsilon) / N_{\text{total}} \propto \exp(-m\upsilon^2 / 2kT)\Delta\upsilon$
  Mean kinetic energy of particles: $m\upsilon^2/2 = 3/2\,kT$
- Boltzmann distribution of energy level population: $N_i \propto \exp(-E_i / kT)$

Up and down

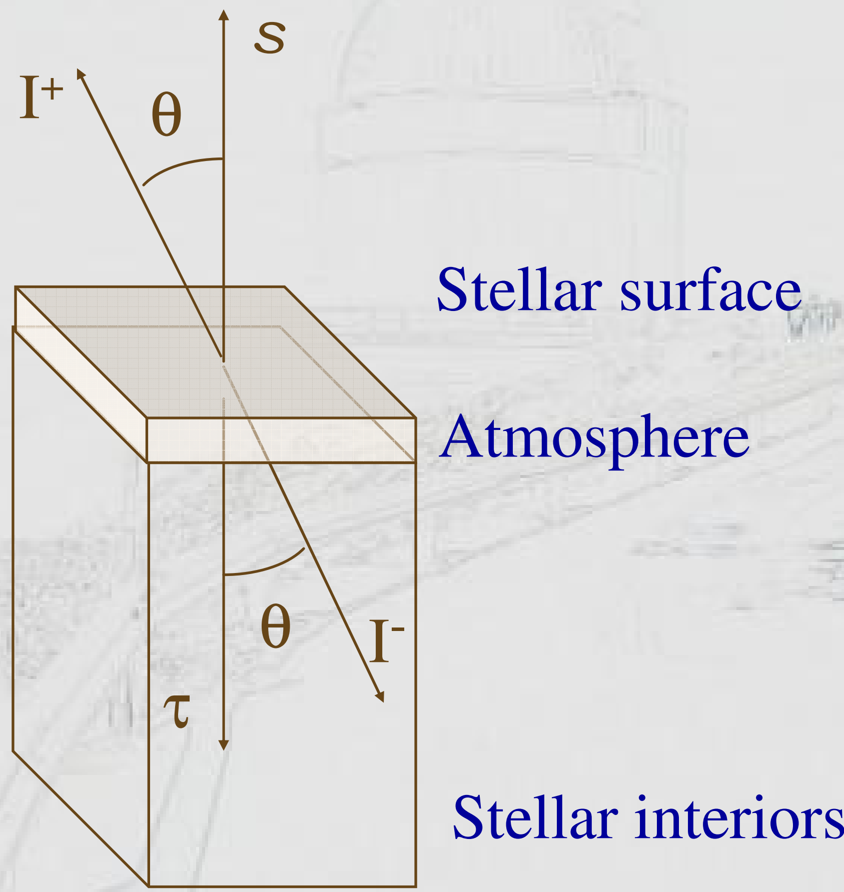
(coordinate system convention)

s - geometrical depth

τ - optical depth, points down

θ- angle between the beam and the normal to the surface. We define $\mu \equiv \cos\theta$, so that the path through the photosphere is $\mu\Delta s$ or $\mu\Delta\tau$

$I^{\pm}$ - are the intensities of the two radiation streams

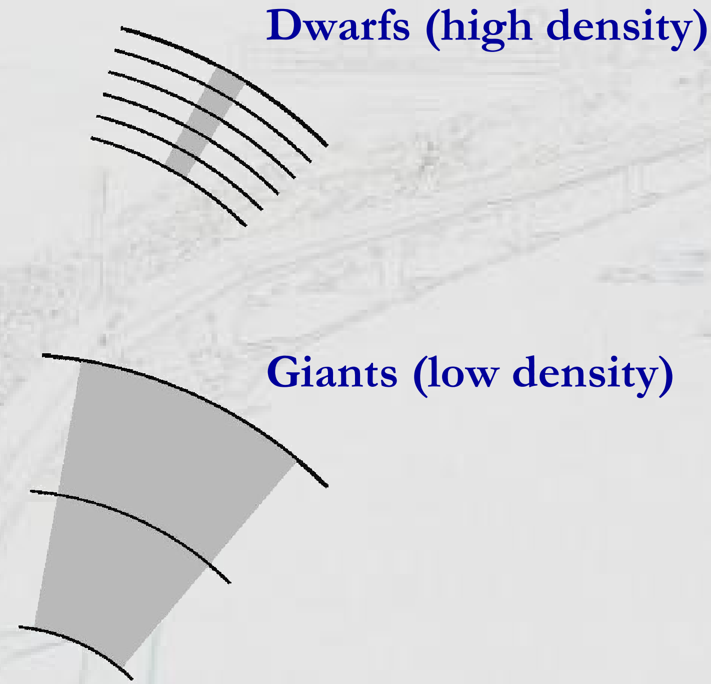


Plane-parallel vs. spherical

All thermodynamic parameters depend only on the radius

Photon mean free path in high and low density atmospheres:

Concentric arcs show radial extent of "visible horizon" while shaded areas mark it tangential size.



Radiation Transport (Units)

- Intensity of radiation: [erg s^{-1} cm^{-2} Hz^{-1} $ster^{-1}$] or [erg s^{-1} cm^{-2} $Å^{-1}$ $ster^{-1}$].
- Radiation transfer equation:

  $$dI_\nu = -k_\nu \rho I_\nu ds + j_\nu ds$$

  J_ν has the units of intensity per cm: [erg s^{-1} cm^{-3} Hz^{-1} $ster^{-1}$].
- One can express emission per unit ***wavelength*** or per unit ***frequency:*** $\Delta E \equiv I_\lambda \Delta\lambda = I_\nu \Delta\nu \Rightarrow I_\lambda = I_\nu \nu^2 / c$
- $k_\nu \rho ds$ must be unitless. Density is [g cm^{-3}] therefore mass extinction must be [cm^2 g^{-1}], while volume extinction is in [cm^{-1}].

- Source function is the ratio of emission to extinction:

$$S_\nu = j_\nu / k_\nu \rho$$

so it has the same units as intensity.

- We can also introduce the concept of optical depth: $d\tau = -k_\nu \rho ds$
- Now the RT equation can be rewritten in canonical form:

$$\mu \frac{dI_\nu}{d\tau} = I_\nu - S_\nu$$

Radiative transfer through simple media:

- No emission: $I_\nu(s) = I_\nu(0)e^{-\tau_\nu(s)}$

  Sounds familiar?

- Optically thick/thin case:

  What is the *mean free path* of a photon in optical depth scale? $\left(\int_0^\infty \tau_\nu e^{-\tau_\nu} d\tau_\nu \Big/ \int_0^\infty e^{-\tau_\nu} d\tau_\nu \right)$

- Formal solution of radiative transfer equation: $I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(t)e^{-(\tau_\nu - t)}dt$

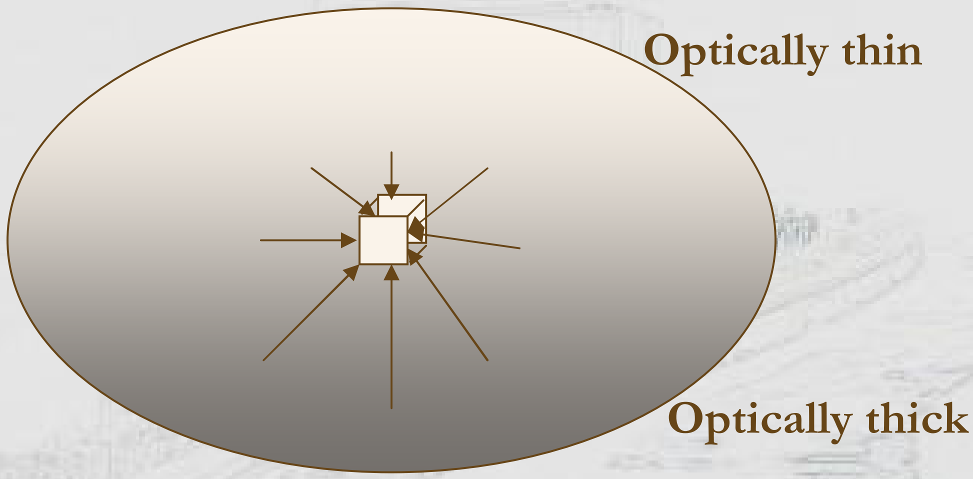
Home work: derive formal solution of the RT equation and solve it for constant source function in a plane-parallel slab

… so why are we so worried about solving the RT equation?

Scattering:

$$S_\nu = \varepsilon B_\nu(T) + (1-\varepsilon)\overline{I_\nu}$$

$$\overline{I_\nu} = \oint I_\nu d\Omega$$



This makes the RT equation non-linear and worst of all, it happens where we least want it: *at the surface*

- The equation of hydrostatic equilibrium (now gravity is constant):

$$\frac{dP_{\mathrm{gas}}}{ds}+\frac{dP_{\mathrm{rad}}}{ds}=-\rho g$$

$$P_{\mathrm{rad}}=\frac{1}{4\pi}\int_{-\pi}^{+\pi} I_{\nu}\cos\theta^{2}\sin\theta d\theta$$

- To solve it we need to know ρ and I_{ν}
- The relation between gas pressure and density involves temperature and mean molecular weight:

$$P_{\mathrm{gas}}=\mathrm{k}\frac{\rho}{\bar{m}}T$$

- The calculation of the mean molecular weight and temperature will involve the solution of equation of state and energy conservation equation.

Steady state

- Static models make things simple but prevent explicit simulations of all dynamic phenomena (convection, waves, winds)
- Model structure (P, T, density and intensity of radiation as function of radius) are established via iterations aiming at constant energy flux at each layer:
  1. Assume density & temperature structure
  2. Solve RT to get energy flux and radiative pressure
  3. Correct temperature to minimize flux variations
  4. Solve hydrostatic equilibrium to get new densities
  5. Repeat from 2 until temperature is constant within few degrees

Local Thermodynamic Equilibrium

- Stellar atmospheres have multiple components (e.g. radiation, gas) characterized by potentially different temperatures. This is because radiation can travel quickly between regions with different temperatures.
  Examples? In which part of Earth's atmosphere this is important?
- **LTE** implies that temperature does not change significantly on a scale of a photon mean free path. This assumption is used for the parts where local density is "high".
- At the surface where density drops quickly we let radiation fly out freely. The temperature for matter is established by heating and cooling rates (absorption and emission of photons). This is called **Radiative Equilibrium**.
- In LTE the source function becomes Planck function $B_\nu(T)$ and the temperature of radiation and gas becomes the same.

Next lecture:

Equation of state and opacities

Comparison with observations

Discussion (tomorrow):

1) *Solve the problems given in the lecture*

2) Think of arguments why the LTE *intensity should have a spectrum of* Plank *function*