Lecture 4:
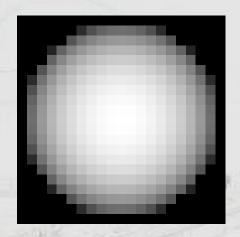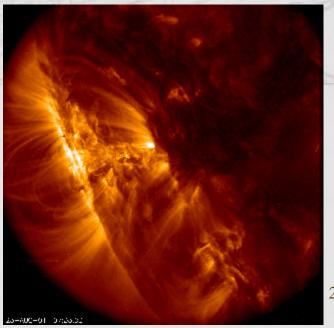## *3D hydrodynamic models*

Motivation

Basically, what we model in 1D is this:

but we really should model is this:

Generalization to 3D dynamical models

- Now we have to solve different problem and different set of time-dependent equations instead of hydrostatic equilibrium

- The equations describe mass, momentum and energy conservation:

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho v_x \\ \rho v_y \\ \rho v_z \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} \rho v_x\, v_x + P \\ \rho v_y\, v_x \\ \rho v_z\, v_x \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} \rho v_x\, v_y \\ \rho v_y\, v_y + P \\ \rho v_z\, v_y \end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} \rho v_x\, v_z \\ \rho v_y\, v_z \\ \rho v_z\, v_z + P \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{\partial}{\partial t}\rho e_{ik} + \frac{\partial}{\partial x}([\rho e_{ik}+P]\, v_x) + \frac{\partial}{\partial y}([\rho e_{ik}+P]\, v_y) + \frac{\partial}{\partial z}([\rho e_{ik}+P]\, v_z) = 0\,.$$

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Coupling with radiation

The equations get two additional terms

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0$$

$$\frac{\partial \rho\boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\cdot\left(\rho\,\boldsymbol{v}{:}\boldsymbol{v} + P\bar{\mathsf{I}} + \bar{\bar{\mathsf{P}}}_{\mathrm{rad}}\right) = 0$$
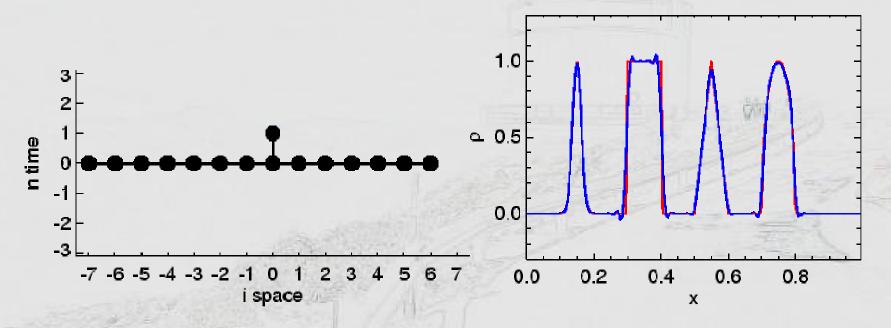
$$\frac{\partial \rho e_{\mathrm{ik}}}{\partial t} + \boldsymbol{\nabla}\cdot\left([\rho\,e_{\mathrm{ik}} + P]\,\boldsymbol{v} + \boldsymbol{F}_{\mathrm{rad}}\right) = 0$$

where the radiative pressure and the radiative energy flux are given by:

$$\bar{\bar{\mathsf{P}}}_{\mathrm{rad}}(\boldsymbol{x}) = \frac{1}{c_{\mathrm{l}}}\int_0^\infty \oint_\Omega \hat{\boldsymbol{n}}{:}\hat{\boldsymbol{n}}\,I_\nu(\boldsymbol{x},\hat{\boldsymbol{n}},\nu)\,d\omega\,d\nu$$

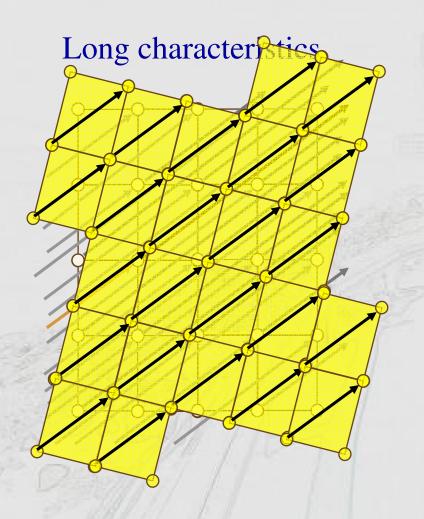$$\boldsymbol{F}_{\mathrm{rad}}(\boldsymbol{x}) = \int_0^\infty \oint_\Omega \hat{\boldsymbol{n}}\,I_\nu(\boldsymbol{x},\hat{\boldsymbol{n}},\nu)\,d\omega\,d\nu$$

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Complex numerical hydro schemes
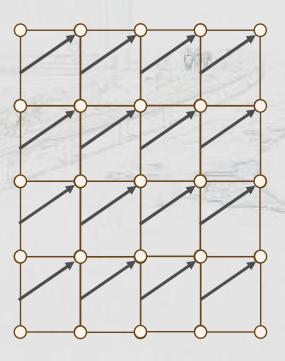
Stencil and example for WENO scheme

RT Solver: Long & Short Characteristics

Long characteristics

Short characteristics

Long characteristics: Feautrier

We define two new variables $U = \frac{1}{2}(I^+ + I^-)$ and $V = \frac{1}{2}(I^+ - I^-)$. Now we can add/subtract the two equations of RT and divide the results by 2:

$$\frac{dI_\nu^+}{dx} = -\alpha_\nu \cdot (I_\nu^+ - S_\nu)$$

$+$

$$\frac{dI_\nu^-}{dx} = \alpha_\nu \cdot (I_\nu^- - S_\nu)$$
___
$$\frac{dU_\nu}{dx} = -\alpha_\nu \cdot V_\nu$$

$$\frac{dI_\nu^+}{dx} = -\alpha_\nu \cdot (I_\nu^+ - S_\nu)$$

$-$

$$\frac{dI_\nu^-}{dx} = \alpha_\nu \cdot (I_\nu^- - S_\nu)$$
___
$$\frac{dV_\nu}{dx} = -\alpha_\nu \cdot (U_\nu - S_\nu)$$

2nd order form of RT

– We substitute the derivative of V in the 2nd equation using the expression for V from the 1st equation:

$$V_v = -\left(\alpha_v\right)^{-1}\frac{dU_v}{dx}$$

$$\frac{dV_v}{dx} = -\alpha_v \cdot (U_v - S_v)$$

– The equations for U and V can be combined to a single 2nd order ODE:

$$\frac{d}{dx}\left[\left(\alpha_v\right)^{-1}\frac{dU_v}{dx}\right] = \alpha_v \cdot (U_v - S_v)$$

Boundary Conditions

Boundary conditions are set in the two ends of the medium. For the smallest x we can write:

$$\left(\alpha_\nu\right)^{-1}\left.\frac{dU_\nu}{dx}\right|_A = -V_\nu = -\frac{1}{2}(I_\nu^+ - I_\nu^-) =$$

$$= \frac{1}{2}(I_\nu^+ + I_\nu^-) - I_\nu^+ = U_\nu - I_\nu^A$$

For the opposite end we have:

$$\left(\alpha_\nu\right)^{-1}\left.\frac{dU_\nu}{dx}\right|_B = -V_\nu = -\frac{1}{2}(I_\nu^+ - I_\nu^-) =$$

$$= -\frac{1}{2}(I_\nu^+ + I_\nu^-) + I_\nu^- = -U_\nu + I_\nu^B$$

Finite differences equation have familiar form (*note the sign change*):

$$-a_i U_{i-1} + b_i U_i - c_i U_{i+1} = d_i \quad \text{for} \quad i = 2, \ldots, N-1$$

$$a_i = \frac{1}{x_{i+1} - x_{i-1}} \cdot \frac{\left(\alpha_{v,i}\right)^{-1} + \left(\alpha_{v,i-1}\right)^{-1}}{x_i - x_{i-1}}$$

$$c_i = \frac{1}{x_{i+1} - x_{i-1}} \cdot \frac{\left(\alpha_{v,i+1}\right)^{-1} + \left(\alpha_{v,i}\right)^{-1}}{x_{i+1} - x_i}$$

$$b_i = a_i + c_i + \alpha_{v,i}$$

$$d_i = \alpha_{v,i} \cdot S_{v,i}$$

Short characteristics: Attenuation Operator

- Solution of RT over one grid cell can be written as:

$$I_\nu(\tau_i) = e^{-(\tau_i - \tau_{i-1})} \cdot I_\nu(\tau_{i-1}) +$$

$$+ \int_{\tau_{i-1}}^{\tau_i} S_\nu(t) \cdot e^{-(\tau_i - t)} dt$$

τ is the optical path along the ray, S is a source function

- If S can be approximated by a polynomial in τ we can take the integral above analytically:

$$I_\nu(\tau_i) = I_\nu(\tau_{i-1}) \cdot e^{-(\tau_i - \tau_{i-1})} +$$

$$+ \alpha_{\nu,i} S_{\nu,i-1} + \beta_{\nu,i} S_{\nu,i} + \gamma_{\nu,i} S_{\nu,i+1}$$

Λ-operator

- Formal solution:
$$I_v(\tau,\mu) = e^{-\tau} \cdot I_v^{\text{bound}} + \int_0^\tau S_v(t,\mu) \cdot e^{-t} \cdot dt$$

- Source function with scattering:
$$S_v(\tau) = \tfrac{1-\varepsilon}{2} \int_{-1}^{1} I_{v'}(\tau,\mu)\, d\mu + \varepsilon\, \mathrm{B}_v(T)$$

- Integral equation for the source function:
$$S_v(\tau) = (1-\varepsilon) \oiint d\Omega \int_0^\tau e^{-(\tau-t)} \cdot S_{v'}(t)\, dt +$$
$$+\varepsilon\, \mathrm{B}_v(T) + (1-\varepsilon) e^{-\tau} \oiint I_v^{\text{bound}}\, d\Omega$$

$$S_v^{\text{bound}}$$

We can re-write this in operator form also known as Λ (lambda) operator

$$\Lambda = \oiint d\Omega \int_0^\tau e^{-(\tau-t)} \cdot dt$$

$$S_\nu(\tau) = (1-\varepsilon)\Lambda S_\nu + \varepsilon B_\nu(T) + (1-\varepsilon) \cdot e^{-\tau} \cdot S_\nu$$

Λ -operator is linear:

$$\Lambda(\alpha \cdot S_1 + \beta \cdot S_2) = \alpha \cdot \Lambda S_1 + \beta \cdot \Lambda S_2$$

Λ-iterations
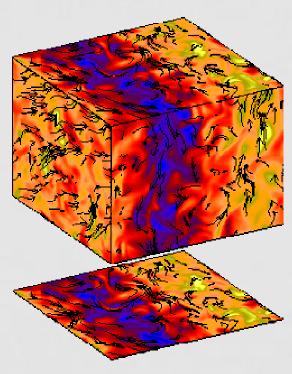
- Recurrency relation:

$$S_0 = \varepsilon \, \mathrm{B}_\nu$$

$$S_{i+1} = (1-\varepsilon)\Lambda S_i + \varepsilon \, \mathrm{B}_\nu + (1-\varepsilon)e^{-\tau}S_\nu^{\mathrm{bound}}$$

- Convergence rate:
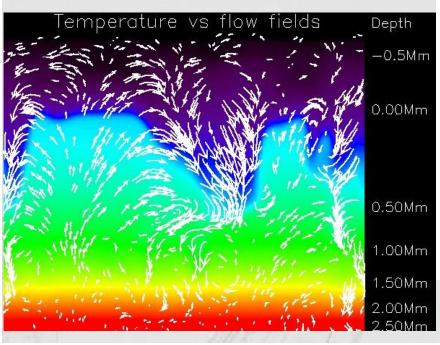
$$\frac{\begin{aligned} S^{n+1} &= (1-\varepsilon)\Lambda S^n + \varepsilon \, \mathrm{B} + (1-\varepsilon)e^{-\tau}S_\nu^{\mathrm{bound}} \\ S^n &= (1-\varepsilon)\Lambda S^{n-1} + \varepsilon \, \mathrm{B} + (1-\varepsilon)e^{-\tau}S_\nu^{\mathrm{bound}} \end{aligned}}{\Delta S^n = (1-\varepsilon)\Lambda(\Delta S^{n-1}) \qquad (\tau > 1)}$$
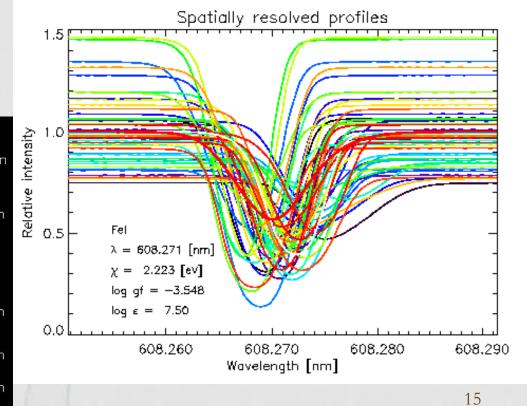
Results

Solar model

Temperature vs flow fields

Depth
- −0.5Mm
- 0.00Mm
- 0.50Mm
- 1.00Mm
- 1.50Mm
- 2.00Mm
- 2.50Mm

Spatially resolved profiles

FeI
$\lambda = 608.271$ [nm]
$\chi = 2.223$ [eV]
$\log gf = -3.548$
$\log \epsilon = 7.50$

Relative intensity

Wavelength [nm]

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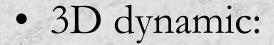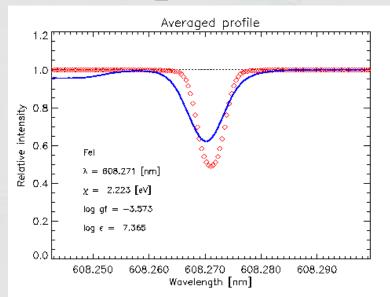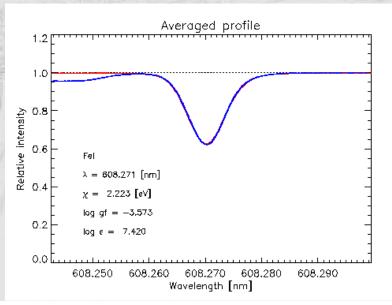
Line shapes

- 1D static:

- 3D dynamic:
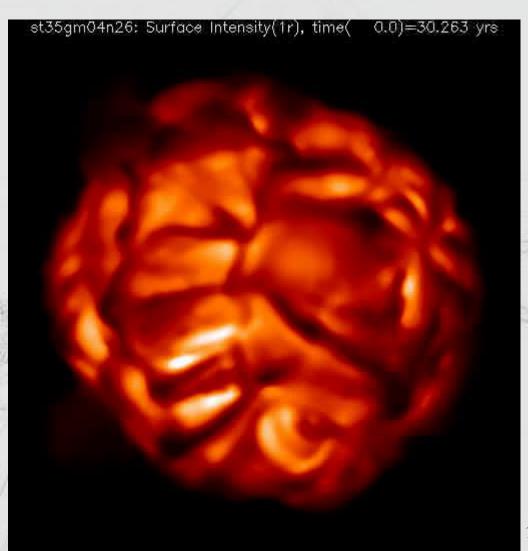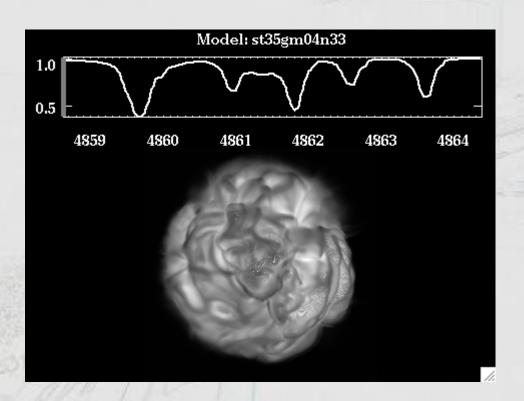
Supergiants: Betelgeuse

Time evolution
of brightness

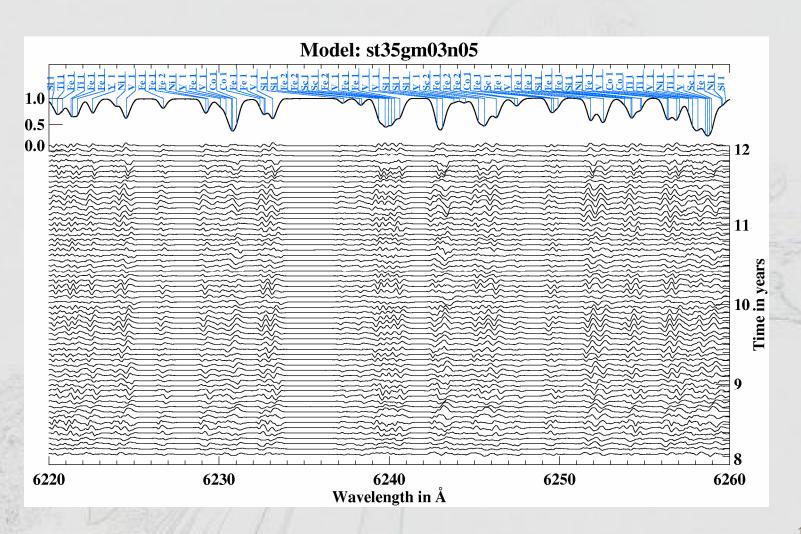st35gm04n26: Surface Intensity(1r), time(0.0)=30.263 yrs

Supergiants: Betelgeuse

Star at different wavelengths:

Synthetic observables

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Evolution of models

- Models started from analytical approach but quickly became numerical because of complexity of physical processes involved: nuclear reactions, radiative energy transport, opacities, convection, turbulence, ionization, magnetic fields, chemistry.

- The numerical models are reaching the next level of realism extending to 3D dynamics and radiative transport. They reproduce observed properties that could not be possibly described in 1D.

- Verification of the models led to more and more sophisticated simulations of observations (synthetic observables) and to the analysis of sensitivity of various observables to stellar properties (e.g. temperature, mass, chemical composition).

- Understanding of such relations is so advanced now that we can formulate and solve inverse problems where we start from observations and derive stellar parameters.

Next lecture:
Doppler Imaging