Introduction to Numerical Hydrodynamics and Radiative Transfer

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Part I: Energy transport by radiation

From static 1D medium to dynamic environment in multi-dimensions

http://www.astro.uu.se/~piskunov/TEACHING/NUM_RT/nh_rt2015.html

The goals of this part:

- *To get* a consistent picture of *radiative energy transport* as part of *hydrodynamics* of complex media using examples of typical astrophysical environment
- *To learn numerical methods* and *approximations* describing hydrodynamics and radiative transfer
- *To understand* the *advantages* and *limitations* of different techniques
- *To obtain* initial *experience in programming* RT and HD by getting acquainted with advanced codes

Course structure and grading

- Structure: The course consists of 2 parts: lectures and a few exercises.

- Grading: In order to complete the course students would have to attend most of the lectures, do all the home work, and successfully complete the basic level exercises in RT and HD. This will result in 15 points.

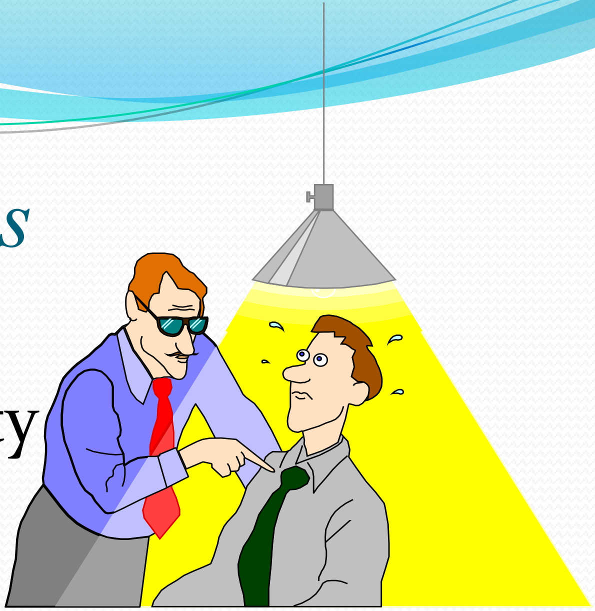
Logical Sequence

- Basics of RT and Numerical Methods
- Numerical Models of Stellar Atmospheres → microphysics → NLTE calculations
- Hydro-dynamical equations → detailed description in 1D → generalization to 3D
- Combining of HD and RT in a single model

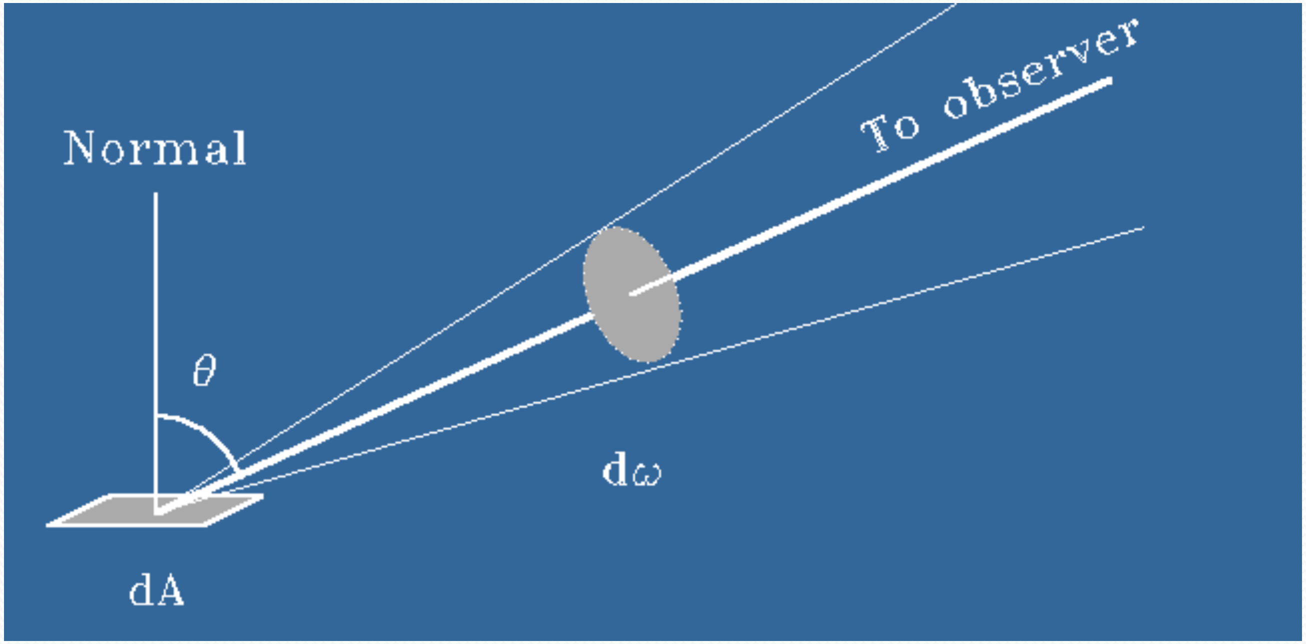
Lecture 1:
Total Recall ***Radiative transfer and basic math***

Radiative transfer: Main Concepts and Definitions

$$\left.\begin{aligned} dE_\nu &= dA\, d\omega\, dt\, d\nu \\ dE_\lambda &= dA\, d\omega\, dt\, d\lambda \end{aligned}\right\}\text{definition of intensity}$$



$I_\lambda d\lambda = I_\nu d\nu$, units I_ν [erg/(s·cm²·rad²·Hz)]



Useful quantities

Intensity	$\Delta E = I_\nu \left[erg/s \cdot cm^2 \cdot Hz \cdot ster \right] \cdot \Delta t \cdot \Delta\sigma \cdot \Delta\nu \cdot \Delta\Omega =$ $= I_\lambda \left[erg/s \cdot cm^2 \cdot \AA \cdot ster \right] \cdot \Delta t \cdot \Delta\sigma \cdot \Delta\lambda \cdot \Delta\Omega$ $I_\nu = I_\lambda \lambda^2/c$
Flux	[erg/s cm^2 Hz]
Mean intensity	$\mathrm{J}_\nu \ \left[\mathrm{erg/s \cdot cm^2 \cdot Hz}\right], \ J_\lambda \ \left[\mathrm{erg/s \cdot cm^2 \cdot \AA}\right]$
Absorption coefficient	$\Delta I_\nu = -\alpha_\nu I_\nu \Delta x = -k_\nu \rho I_\nu \Delta x$ $\alpha_\nu \equiv \alpha_\lambda \left[cm^{-1}\right]; \quad k_\nu \equiv k_\lambda \left[cm^2 \cdot g^{-1}\right]$
Emission coefficient	$\Delta E = j_\nu^{\mathrm{Vol}} \left[erg/s \cdot cm^3 \cdot Hz \cdot ster \right] \cdot \Delta t \cdot \Delta \mathrm{V} \cdot \Delta\nu \cdot \Delta\Omega =$ $= j_\nu^{\mathrm{mass}} \left[erg/s \cdot g \cdot Hz \cdot ster \right] \cdot \rho \cdot \Delta t \cdot \Delta \mathrm{V} \cdot \Delta\nu \cdot \Delta\Omega$
Optical path	$d\tau_\nu = \alpha_\nu(x) \cdot dx = k_\nu(x) \cdot \rho \cdot dx$ [unitless!]
Source function	$S_\nu = j_\nu / k_\nu$ [units of intensity!]

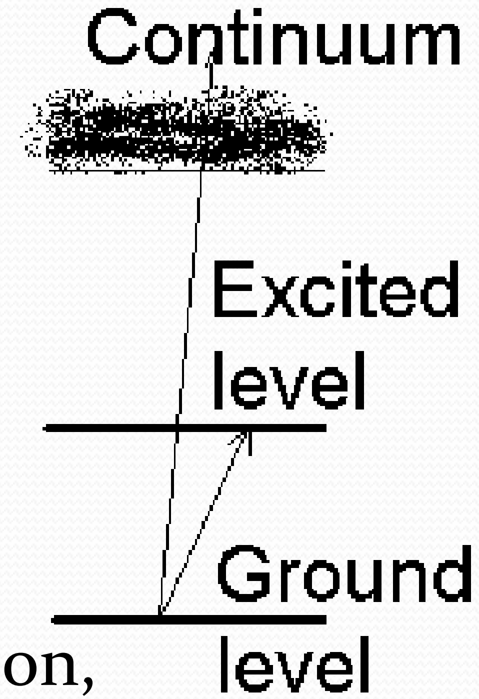
- Absorption and emission contain the "true" part (*energy is transferred between the kinetic energy of the gas and the radiation field*) and the scattering part (*energy of absorbed photon which is returned to the radiation field*).
- <u>Radiation-dominated gas</u>: pure scattering.
- <u>Collision-dominated gas</u>: pure absorption.
- In general case:

$$\kappa_\lambda = k_\lambda + \sigma_\lambda,\ S_\lambda = S_\lambda^A + S_\lambda^S$$

- For isotropic scattering and thermal equlibrium:

$$S_\nu = \frac{\sigma_\nu}{k_\nu + \sigma_\nu} J_\nu + \frac{k_\nu}{k_\nu + \sigma_\nu} B_\nu$$

- One can distinguish 3 types of absorption processes:
- ***b-b*** - radiative transitions
  - collisional transitions
- ***b-f*** - ionization and recombination
- ***f-f*** -absorption/emission



- <u>Radiative</u> ***b-b*** transitions: absorption, spontaneous and stimulated emission.
- <u>Collisional</u> ***b-b*** transitions: excitation and de-excitation

- Equation of radiative transfer connects the change in intensity along a ray as function of absorption and emission:

$$dI_\nu = -\kappa_\nu \rho I_\nu dx + j_\nu \rho dx$$

or

$$\frac{dI_\nu}{d\tau_\nu} = -I_\nu + S_\nu$$

- The formal solution (home work 1a: *derive this*):

$$I_\nu(\tau_\nu'') = I_\nu(\tau_\nu') \cdot e^{-(\tau_\nu'' - \tau_\nu')} + \int_{\tau_\nu'}^{\tau_\nu''} S_\nu(t) e^{-(\tau_\nu'' - t)} dt$$

Critical dependencies

- Geometrical, angular, frequency dependence of opacity k_ν and source function S_ν
- Dependence of the source function S_ν on the radiation field
- Number of absorbers (how many absorbers there is on a given energy level) depend on local physical conditions and radiation field
- Velocity distribution of the absorbers affects the frequency dependence of κ_ν and S_ν

Home work 1b: think of some where one of the above has dramatic effects on radiation field

Our initial approximations

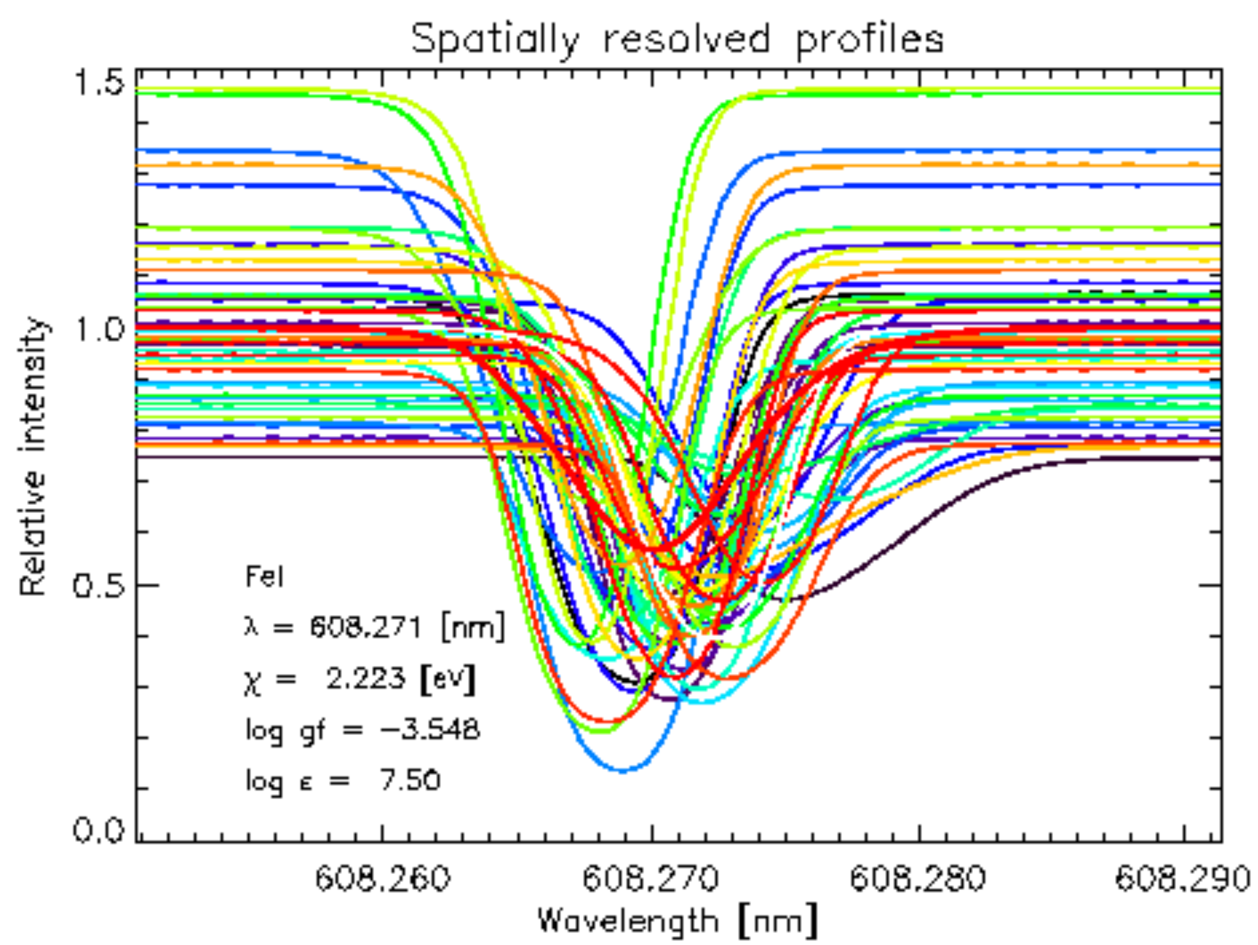
- Time: No time dependence (static)
- Geometry: plane-parallel or spherical medium
- Isotropy at any depth
- Absorbers: Boltzmann level population, Saha ionization balance, Maxwellian velocity distribution
- Line shapes: identical absorption and emission profiles ≡ Voigt profile
- Local Thermodynamic Equilibrium (LTE, how good is it?)

We will gradually drop some of the assumptions

Examples

- Photospheres of solar-like stars (convection)
- Giant stars (spherical, anisotropic radiation field, giant convectiv cells)
- Stellar winds (complex geometry, velocity field, anisotropic radiation field, NLTE, dynamic)
- Gas clouds (LTE?, external radiation field, different T_{rad} and T_{gas}, presence of dust)

Solar convection and emerging spectra



Courtesy of Martin Asplund

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Convection on Betelgeuse

Courtesy of Bernd Freytag

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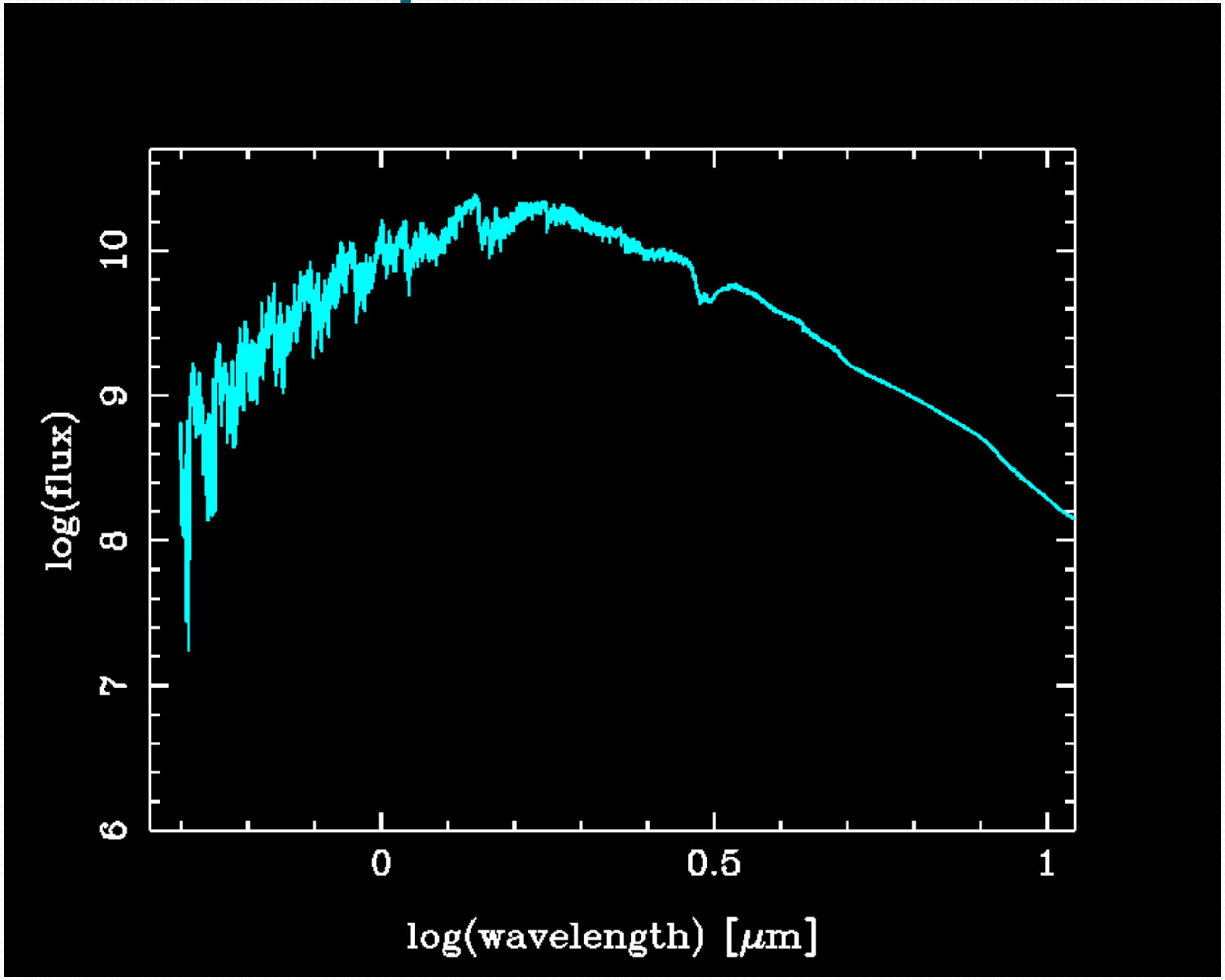
Dynamic spectra



Courtesy of Susanne Höfner

Examples

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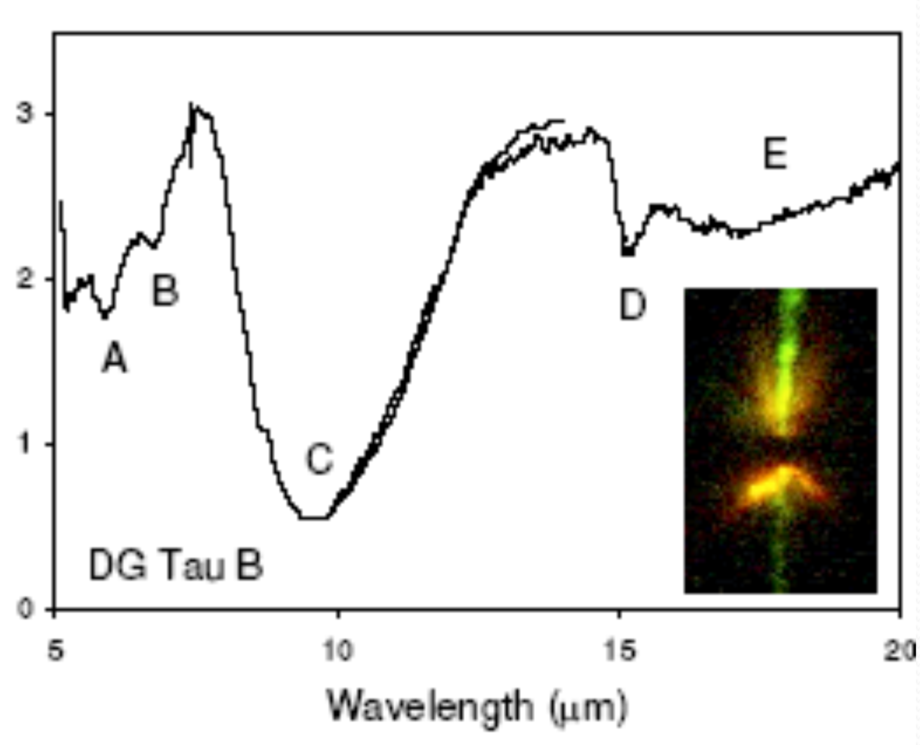
Protostars



A – water ice

B – methanol ice

C & E – amorphous silicates

D – carbon-dioxide ice

Watson et al.: 2004, Astrophysical Journal Supp Series 154, 391

Next part

- Math: first and second order ordinary differential equations, partial differential equations, boundary conditions, direct integration schemes, finite differences, convergence and stability, vector ODE. Gauss quadratures, solving systems of linear equations. Non-linear equations.
- Press et al. “Numerical Recipes: The Art of Scientific Computing”

Total Recall

Math, part 1

System of Linear Equations (SLE):

$$
\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1N}x_N = b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2N}x_N = b_2 \\
\qquad\qquad\qquad \ldots \qquad\qquad \ldots \\
a_{N1}x_1 + a_{N2}x_2 + \ldots + a_{NN}x_N = b_N
\end{array}
$$

$$\mathbf{A \cdot x = b}$$

Two important algorithms: *Gauss-Jordan elimination* and *LU decomposition*

Gauss-Jordan Elimination

Simple rules:

i. Changing places of any two rows in **A** requires only a similar change in **b**.
ii. Replacing any row in **A** and **b** with a linear combination of itself and other rows does not change the solution.
iii. Interchanging two columns in **A** is equivalent to changing the sequence of **x**, therefore the solution must be sorted to get the original sequence.

No pivoting

$$a_{11}x_1+a_{12}x_2+ \quad \dots = b_1$$
$$a_{21}x_1+a_{22}x_2+ \quad \dots = b_2$$
$$a_{31}x_1+a_{32}x_2+ \quad \dots = b_3$$

$$a_{11}x_1+a_{12}x_2+ \quad \dots =b_1$$
$$0+ \quad (a_{22}-a_{12}a_{21}/a_{11})x_2+\dots =b_2-b_1a_{21}/a_{11}$$
$$0+ \quad (a_{32}-a_{12}a_{31}/a_{11})x_2+\dots =b_3-b_1a_{31}/a_{11}$$

$$a'_{11}x_1+a'_{12}x_2+ \quad \dots = b'_1$$
$$a'_{22}x_2+ \quad \dots = b'_2$$
$$a'_{32}x_2+ \quad \dots = b'_3$$

Partial pivoting

$$a_{11}x_1+a_{12}x_2+ \quad \ldots = b_1$$
$$a_{21}x_1+a_{22}x_2+ \quad \ldots = b_2$$
$$a_{31}x_1+a_{32}x_2+ \quad \ldots = b_3$$

k: $|a_{k1}|=\max(|a_{j1}|)$ for j=i,i+1,… N

$$a_{k1}x_1+a_{k2}x_2+ \quad \ldots =b_k$$
$$0+ \quad (a_{22}-a_{k2}a_{21}/a_{k1})x_2+\ldots =b_2-b_ka_{21}/a_{k1}$$
$$0+ \quad (a_{32}-a_{k2}a_{31}/a_{k1})x_2+\ldots =b_3-b_ka_{31}/a_{k1}$$

$$a'_{11}x_1+a'_{12}x_2+ \quad \ldots = b'_1$$
$$a'_{22}x_2+ \quad \ldots = b'_2$$
$$a'_{32}x_2+ \quad \ldots = b'_3$$

Full pivoting

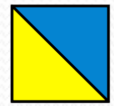
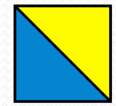
$$a_{11}x_1+a_{12}x_2+ \quad \ldots = b_1$$
$$a_{21}x_1+a_{22}x_2+ \quad \ldots = b_2$$
$$a_{31}x_1+a_{32}x_2+ \quad \ldots = b_3$$

k,l: $|a_{kl}|=\max(|a_{nm}|)$ for n,m=i,i+1,… N

$$a_{k1}x_1+a_{k2}x_2+ \quad \ldots =b_k$$
$$0+ \quad (a_{22}-a_{k2}a_{21}/a_{k1})x_2+\ldots =b_2-b_k a_{21}/a_{k1}$$
$$0+ \quad (a_{32}-a_{k2}a_{31}/a_{k1})x_2+\ldots =b_3-b_k a_{31}/a_{k1}$$

$$a'_{11}x_1+a'_{12}x_2+ \quad \ldots = b'_1$$
$$a'_{22}x_2+ \quad \ldots = b'_2$$
$$a'_{32}x_2+ \quad \ldots = b'_3$$

LU decomposition

A = L·U where **L** and **U** are triangular matrices **L:**  **U:** 

$\mathbf{A \cdot x = (L \cdot U) \cdot x = L \cdot (U \cdot x) = b}$

$\mathbf{L \cdot y = b}$ and $\mathbf{U \cdot x = y}$

Solving these systems is easy:

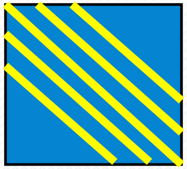
$\mathbf{y_1 = b_1 / L_{11}; \quad y_2 = (b_2 - y_1 \cdot L_{21}) / L_{22}}$ etc.

See Num. Rec. Section 2.3 on how to compute **L** and **U**.

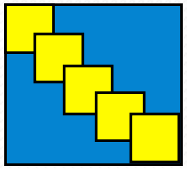
<u>Home work 1c</u>: Write the algorithm for constructing L and U and explain why LU is faster than GJ

Special matrices

- Tri-diagonal: forward and back-substitution, (no difference between Gauss-Jordan and LU decomposition schemes)
- Band-diagonal



- Block-diagonal



Iterative improvement of the solution:

$$\mathbf{A}\cdot(\mathbf{x}'-\mathbf{x}) = \mathbf{A}\cdot\mathbf{x}' - \mathbf{b}$$