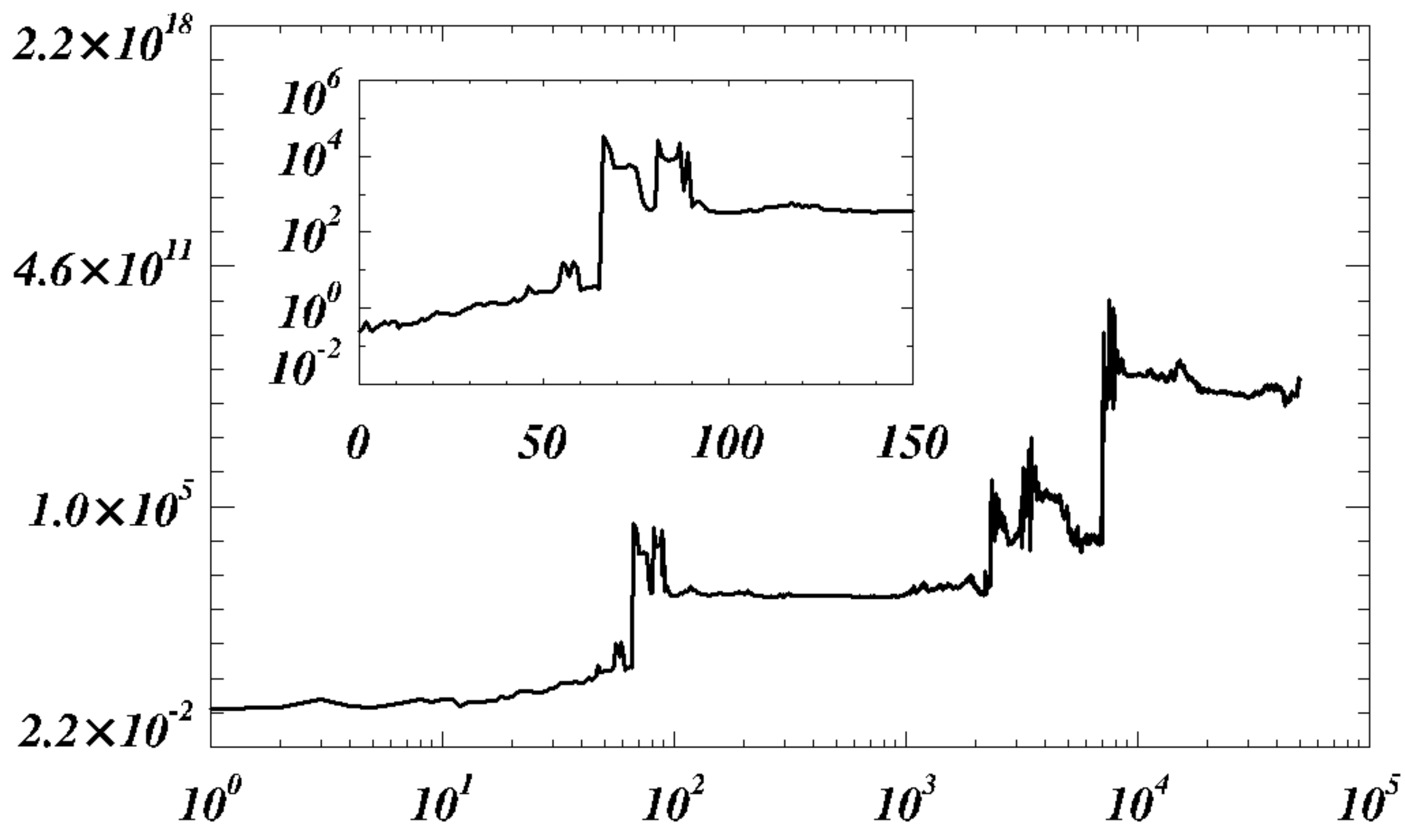
Home Work #1 (problem 1)

Inversion results:



HW #1 (problem 3)

Analysis of the equation:

$$\frac{dy}{dx} = y + e^x; \qquad y\Big|_{x=0} = 1; \qquad y\Big|_{x=50} = 1000;$$

$$y = (\tfrac{1}{2} \cdot x - \tfrac{1}{4} + a) \cdot e^x + b \cdot e^{-x}$$

$$\begin{cases} a + b = \tfrac{5}{4} \\ a + b \cdot e^{-100} = 1000 \cdot e^{-50} - \tfrac{99}{4} \end{cases} \xrightarrow{\approx} \begin{cases} a = 1000 \cdot e^{-50} - \tfrac{99}{4} \\ b = \tfrac{104}{4} \end{cases}$$

RK4 with adaptive step

- System of ODE:

$$\frac{dy_1}{dx} = y_2; \qquad \frac{dy_2}{dx} = y_1 + e^x$$

- Aiming parameter (AP):

$$\{y_1, y_2\}\big|_{x=0} = \{1, \text{ aim high } (\, y_1\big|_{\mathrm{x=50}} > 1000)\}$$

$$\{y_1, y_2\}\big|_{x=0} = \{1, \text{ aim low } (\, y_1\big|_{\mathrm{x=50}} < 1000)\}$$

- Problem: AP is ≈2, change in AP of 10^{-12} changes $y_1(50)$ by 10^{20}

Solutions

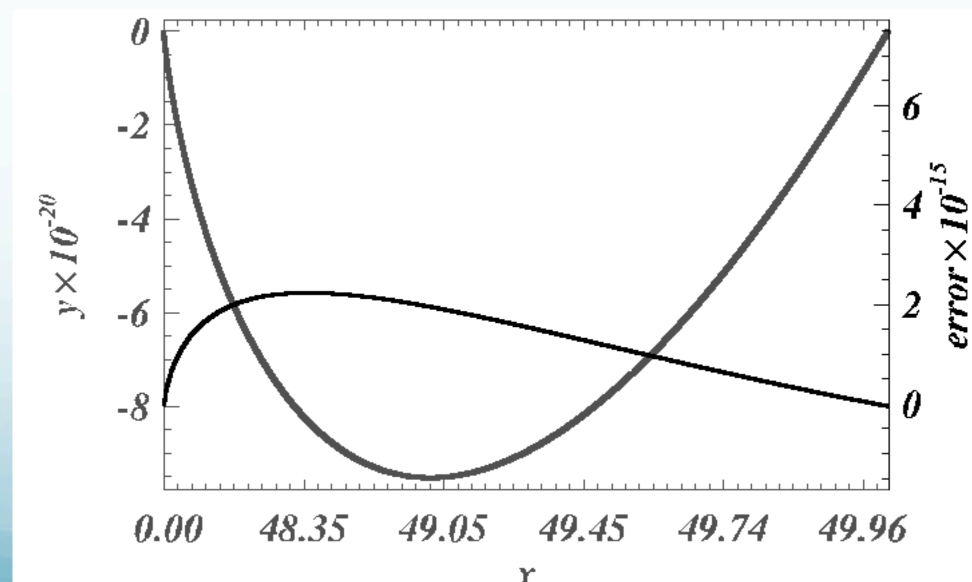
- *Algorithm A*:
  - Aim from x=0
  - Aim from x=50
  - Take $y = (50 - x) / 50 \cdot y^{\text{left}} + x/50 \cdot y^{\text{right}}$
- *Algorithm B*: replace variable $t = e^{x}$

Finite differences

Straight forward:

$$\begin{bmatrix} 1 & & & & 0 \\ & \ddots & & & \\ 0 & 1 & -2-\Delta^2 & 1 & 0 \\ & & & \ddots & \\ 0 & & & & 1 \end{bmatrix} \times \begin{bmatrix} y_1 \\ \vdots \\ y_k \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} y^{\text{left}} \\ \vdots \\ \Delta^2 e^{k\Delta} \\ \vdots \\ y^{\text{right}} \end{bmatrix}$$

Solution:



Lecture 4
Self-consistent solution of RT

- Taking into account the intensity part of the source function (scattering)
- Constructing self-consistent solution for the intensity and the source function (Λ-iterations)
- Assessing the efficiency of the process in different situations
- Speeding things up

Source function again

Source function is a ratio of emission to absorption:

$$S_\nu(x,\mu) = \frac{j_\nu(x,\mu)}{k_\nu(x,\mu)}$$

The emission consists of two parts: *stimulated* ($\propto$ to local intensity) and *spontaneous* (function of the local conditions):

$$S_\nu(x,\mu) = \frac{j_\nu^{\text{stim}}(x,\mu) + j_\nu^{\text{spont}}(x,\mu)}{k_\nu(x,\mu)} =$$

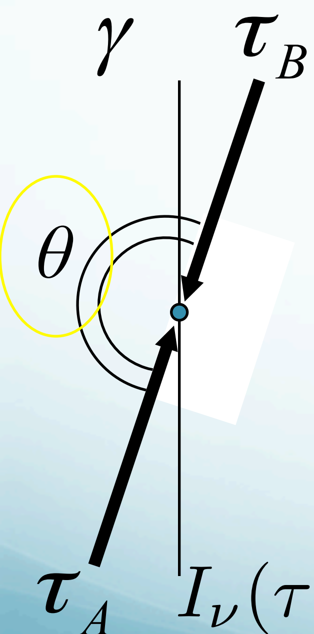
$$= (1-\varepsilon)\cdot J_\nu(x) + \varepsilon \cdot B_\nu$$

RT for a *b-b* transition

A complete set of equations for a single *b-b* transition:

$$\frac{dI_\nu(\tau,\gamma,\theta)}{d\tau} = -I_\nu(\tau,\gamma,\theta) + S_\nu(\tau,\gamma,\theta)$$

$$S_\nu(\tau,\gamma,\theta) = \frac{1-\epsilon}{4\pi}\cdot\int\limits_{-\infty}^{\infty}\phi(\nu-\nu')d\nu'\oint I_\nu d\gamma d\theta + \epsilon\cdot B_\nu(T)$$



Now we can write the formal solution:

$$I_\nu(\tau,\gamma,\theta) = \begin{cases} \int\limits_{\tau}^{\tau_B} S_\nu(t,\gamma,\theta)\cdot e^{-(t-\tau)}dt + e^{-(\tau_B-\tau)}\cdot I_\nu^B \\ \int\limits_{\tau_A}^{\tau} S_\nu(t,\gamma,\theta)\cdot e^{-(\tau-t)}dt + e^{-(\tau-\tau_A)}\cdot I_\nu^A \end{cases}$$

Formal solutions: coordinate convention

- Source function:

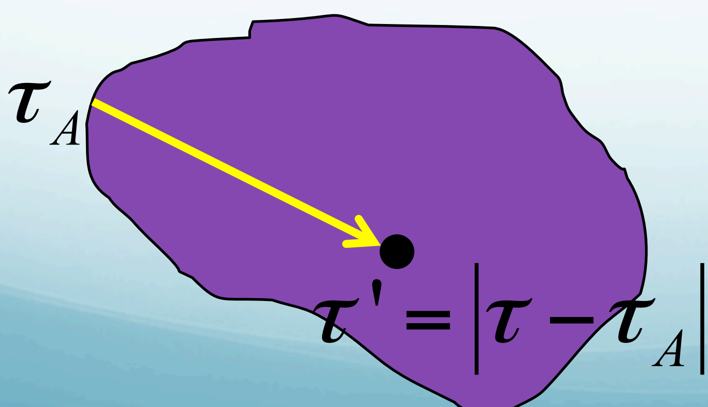
$$S_\nu(\tau,\gamma,\theta) = \frac{1-\epsilon}{4\pi}\cdot\int\limits_{-\infty}^{\infty}\phi(\nu-\nu')d\nu'\oint I_\nu d\gamma d\theta + $$

$$+ \quad \epsilon\cdot B_\nu(T)$$

- Intensity:

$$I_\nu(\tau,\gamma,\theta) = e^{-\tau'}\cdot I_\nu^A +$$

$$+ \int\limits_0^{\tau'} S_\nu(t,\gamma,\theta)\cdot e^{-t}dt$$



Now we can combine formal solutions in one self-consistent integral equation:

$$S_\nu(\tau) = (1-\epsilon)\oint \frac{d\Omega}{4\pi}\int_o^\tau \int_{-\infty}^{\infty} \phi(\nu-\nu')\cdot e^{-(\tau-t)}\cdot S_{\nu'}(t)d\nu'$$

$$+ \quad (1-\epsilon)e^{-\tau}\oint \frac{d\Omega}{4\pi}\int_{-\infty}^{\infty} \phi(\nu-\nu')\cdot I_{\nu'}^{\text{bound}} d\nu' + \epsilon B_\nu(T)$$

S_ν^{bound}

We can re-write this in operator form also known as Λ (lambda) operator

$$\Lambda = \oint \frac{d\Omega}{4\pi}\int_o^\tau \int_{-\infty}^{\infty} \phi(\nu-\nu')\cdot e^{-(\tau-t)}\cdot dt\cdot d\nu'$$

$$S_\nu(\tau) = (1-\epsilon)\Lambda S_\nu + \epsilon B_\nu(T) + (1-\epsilon)\cdot e^{-\tau}\cdot S_\nu^{\text{bound}}$$

Λ -operator is linear:

$$\Lambda(\alpha\cdot S_1 + \beta\cdot S_2) = \alpha\cdot\Lambda S_1 + \beta\cdot\Lambda S_2$$

Λ-iterations

- Recurrent relation:
- $S_0 = \varepsilon \mathrm{B}_\nu$

$$S_{i+1} = (1-\varepsilon)\Lambda S_i + \varepsilon \mathrm{B}_\nu + (1-\varepsilon)e^{-\tau} S_\nu^{\text{bound}}$$

- Convergence rate:

$$S^{n+1} = (1-\varepsilon)\Lambda S^n + \varepsilon \mathrm{B} + (1-\varepsilon)e^{-\tau} S_\nu^{\text{bound}}$$

$$-$$

$$S^{n} = (1-\varepsilon)\Lambda S^{n-1} + \varepsilon \mathrm{B} + (1-\varepsilon)e^{-\tau} S_\nu^{\text{bound}}$$

$$\Delta S^n = (1-\varepsilon)\Lambda(\Delta S^{n-1}) \qquad (\tau > 1)$$

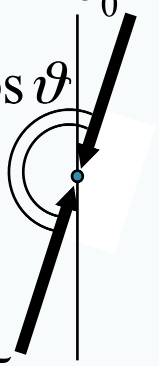
Convergence etc.

- Convergence is slow for small ε and τ >1.

$$\Delta S^n = (1-\varepsilon)\Lambda(\Delta S^{n-1})$$

- Convergence is monotonous (corrections have always the same sign).
- Note that: $\Lambda S_\nu = J_\nu$
- We could also substitute the expression for the source function to the expression for intensity. This is useful when we need to include the radiative energy transport to hydrodynamics.
- Home work: *write the expression for* Λ *-operator for angle averaged intensity* J_ν

1D semi-infinite medium



Given one selected direction:

$$\mu \frac{dI_\nu(\tau,\mu)}{d\tau} = I_\nu(\tau,\mu) - S_\nu(\tau)$$

$$S_\nu(\tau,\mu) = \tfrac{1-\varepsilon}{2} \int_{-\infty}^{\infty} \varphi(\nu-\nu')d\nu' \int_{-1}^{1} I_{\nu'}(\tau,\mu)d\mu + \varepsilon\, \mathrm{B}_\nu(T)$$

Now we can write the formal solution:

$$I_\nu(\tau,\mu) = \begin{cases} \frac{1}{\mu}\int_{\tau}^{\infty} s_\nu(t)\cdot e^{-\frac{(t-\tau)}{\mu}}dt, & \mu > 0 \\ -\frac{1}{\mu}\int_{0}^{\tau} s_\nu(t)\cdot e^{-\frac{(t-\tau)}{\mu}}dt, & \mu < 0 \end{cases}$$

Now we can write a self consistent equation for the source function in 1D:

$$S_\nu(\tau) = \tfrac{1-\varepsilon}{2}\left[\int_0^1 \frac{d\mu}{\mu}\int_\tau^\infty \int_{-\infty}^\infty \varphi(\nu-\nu')d\nu' \cdot e^{-\frac{(t-\tau)}{\mu}} dt \cdot S_{\nu'}(t) - \right.$$

$$\left.\int_{-1}^0 \frac{d\mu}{\mu}\int_0^\tau \int_{-\infty}^\infty \varphi(\nu-\nu')d\nu' \cdot e^{-\frac{(t-\tau)}{\mu}} dt \cdot S_{\nu'}(t)\right] + \varepsilon\,\mathrm{B}_\nu(T)$$

$$\Lambda = \tfrac{1}{2}\left[\int_0^1 \frac{d\mu}{\mu}\int_\tau^\infty \int_{-\infty}^\infty \varphi(\nu-\nu')d\nu' \cdot e^{-\frac{(t-\tau)}{\mu}} dt - \right.$$

$$\left.\int_{-1}^0 \frac{d\mu}{\mu}\int_0^\tau \int_{-\infty}^\infty \varphi(\nu-\nu')d\nu' \cdot e^{-\frac{(t-\tau)}{\mu}} dt\right]$$

$$S_\nu(\tau) = (1-\varepsilon)\Lambda S_\nu + \varepsilon\,\mathrm{B}_\nu(T)$$

Accelerated Λ-iterations

- Λ-operator is a convolution with non-negative kernel in frequency and spatial domains. Numerically it can be approximated with summation:

$$\Lambda S_\nu \approx \sum_{\nu,\tau} \omega_{\nu,\tau} s_\nu(\tau) = \sum_k \omega_{j,k} s_k$$

- Newton-Raphson acceleration (Cannon)

$$S - (1-\varepsilon)\Lambda S - \varepsilon B = \Phi(S) = 0$$

- Operator splitting (Scharmer, Olson et al.)

$$S^{(l+1)} = (1-\varepsilon)\Lambda S^{(l)} + \varepsilon B$$

$$\Lambda = \Lambda^* + (\Lambda - \Lambda^*)$$

The recurrent relation for lambda iterations (Cannon):

$$S_\nu^{(l+1)}(\tau) - (1-\varepsilon)\Lambda S^{(l)} - \varepsilon \mathrm{B}_\nu(\tau) = 0$$

$$S_j^{(l+1)} = (1-\varepsilon_j)\sum_{k=1}^{N} \omega_{jk} S_k^{(l)} + \varepsilon_j B_j$$

$$\Delta S_j^{(l)} = (1-\varepsilon_j)\sum_{k=1}^{N} \omega_{jk} S_k^{(l)} + \varepsilon_j B_j - S_j^{(l)} = \Phi(S_k^{(l)})$$

$$S_j^{(l+1)} = S_j^{(l)} - \frac{\Phi(S_k^{(l)})}{d\Phi(S_k^{(l)}) \big/ dS_j^{(l)}} =$$

$$= \frac{(1-\varepsilon_j)\sum_{k\neq j}^{N} \omega_{jk} S_k^{(l)} + \varepsilon_j B_{\nu j}}{(1-\varepsilon_j)\omega_{jj} - 1}$$

Operator splitting (Scharmer, Olson):

$$S^{(l)} + \delta S^{(l)} = (1-\varepsilon)\Lambda(S^{(l)} + \delta S^{(l)}) + \varepsilon\,\mathrm{B}$$

$$\delta S^{(l)} \equiv S^{(l+1)} - S^{(l)}$$

$$\Lambda = \Lambda^{*} + (\Lambda - \Lambda^{*})$$

$$\begin{aligned} S^{(l)} + \delta S^{(l)} = {} & (1-\varepsilon)\Lambda^{*}S^{(l)} + \underline{(1-\varepsilon)\Lambda^{*}\delta S^{(l)}} + \\ & + \underline{(1-\varepsilon)\Lambda S^{(l)}} + \underline{\underline{(1-\varepsilon)\Lambda\delta S^{(l)}}} - \\ & - (1-\varepsilon)\Lambda^{*}S^{(l)} - \underline{\underline{(1-\varepsilon)\Lambda^{*}\delta S^{(l)}}} + \varepsilon\,\mathrm{B} \end{aligned}$$

$$\begin{aligned} S^{(l)} + \delta S^{(l)} = {} & (1-\varepsilon)\Lambda^{*}\delta S^{(l)} + (1-\varepsilon)\Lambda S^{(l)} + \\ & \underline{\underline{+ (1-\varepsilon)[\Lambda - \Lambda^{*}]\delta S^{(l)}}} + \varepsilon\,\mathrm{B} \end{aligned}$$

$$\delta S^{(l)} - (1-\varepsilon)\Lambda^{*}\delta S^{(l)} \approx (1-\varepsilon)\Lambda S^{(l)} + \varepsilon\,\mathrm{B} - S^{(l)}$$

$$\delta S^{(l)} = [1-(1-\varepsilon)\Lambda^{*}]^{-1}\cdot[(1-\varepsilon)\Lambda S^{(l)} + \varepsilon\,\mathrm{B} - S^{(l)}]$$

More general assessment of what was done:

- Λ-iterations are used to find source function in case of radiation-dependent absorption/emission
- The direct iteration scheme for the source function can be used but the convergence is slow, especially at high optical depths
- Lambda integral in ν and τ can be replaced with a quadrature formula using a discrete grid
- We have replaced the true lambda operator with an approximate operator that has better convergence properties (higher order convergence) and can be easily inverted
- The freedom in choice of the accelerated lambda operator helps to optimize it for special cases