# Lecture 5:

## Formal solvers of the RT equation

# Formal RT solvers

- Runge-Kutta (reference solver)
  *Piskunov N.: 1979, Master Thesis*

- Long characteristics (Feautrier scheme)
  *Cannon C.J.: 1970, ApJ 161, 255*

- Short characteristics (Hermitian scheme)
  *Bellot Rubio et al.: 1998, ApJ 506, 805*

- Short characteristics (Bezier attenuation operator)
  *de la Cruz Rodríguez & Piskunov: 2013, ApJ 764, 33*

# Why we need a formal solver?

- What is a formal solver? We assume that we know the source function and the opacities along our ray. This is sufficient to compute the intensities (as function of wavelength).

- In practice, local opacities and source function may also depend on the intensities coming from different directions. This will require iterations.

- In the next lecture we will talk about how to get to self-consistency.

# Solving RT with RK

Simple minded approach:

$$\frac{dy}{dx} = -f(x)\cdot y + g(x); \quad y(x_0) = y_0$$

$$k_1 = -f(x_i)\cdot y_i + g(x_i)$$

$$k_2 = -f\left(x_i + \frac{h}{2}\right)\cdot\left(y_i + \frac{hk_1}{2}\right) + g\left(x_i + \frac{h}{2}\right)$$

$$k_3 = -f\left(x_i + \frac{h}{2}\right)\cdot\left(y_i + \frac{hk_2}{2}\right) + g\left(x_i + \frac{h}{2}\right)$$

$$k_4 = -f(x_i + h)\cdot(y_i + hk_3) + g(x_i + h)$$

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

More clever RK. Previous example suffers from all problems inherent to RK, specially when dealing with complex medium where *f* and *g* have sharp variations. Instead one can solve RT analytically:

$$I_\nu(\tau''_\nu) = I_\nu(\tau'_\nu)\cdot e^{-\Delta\tau_\nu} + \int_{\tau'_\nu}^{\tau''_\nu} B_\nu(t)\cdot e^{-(\tau''_\nu - t)}\,dt$$

In particular, this is useful for a half-infinite medium where we can easily use Gauss quadratures for the integral:

$$I_\nu(0) = \int_0^\infty B_\nu(t)\cdot e^{-t}\,dt = \sum_{i=1}^{N} \omega_i \cdot B_\nu(\tau_{\nu,i})$$

## The nodes and weights for Laguerre polynomials:

Nodes | Weights

| Nodes | Weights |
|---|---|
| 0.137793470540 | 3.08441115765E-01 |
| 0.729454549503 | 4.01119929155E-01 |
| 1.808342901740 | 2.18068287612E-01 |
| 3.401433697855 | 6.20874560987E-02 |
| 5.552496140064 | 9.50151697518E-03 |
| 8.330152746764 | 7.53008388588E-04 |
| 11.843785837900 | 2.82592334960E-05 |
| 16.279257831378 | 4.24931398496E-07 |

The only problem is that values of ***T*** are not known in $\tau_{\nu,\iota}$ We can find them solving ODE for optical depth:

$$\frac{dx}{d\tau_\nu} = \frac{1}{k_\nu(x)\cdot\rho(x)}, \qquad x\Big|_{\tau_\nu=0} = 0$$

Advantages: *simple boundary condition, RHS does not depend on unknown function and RHS is always non-negative.*

## 4th order Runge-Kutta for the geometrical depth

$$\frac{dx}{d\tau_\nu} = \frac{1}{\alpha_\nu(x)}; \qquad x_0 = 0$$

$$k_1 = \frac{1}{\alpha_\nu(x_i)}; \qquad k_2 = \frac{1}{\alpha_\nu\left(x_i + hk_1/2\right)}$$

$$k_3 = \frac{1}{\alpha_\nu\left(x_i + hk_2/2\right)}; \quad k_4 = \frac{1}{\alpha_\nu\left(x_i + hk_3\right)}$$

$$x_{i+1} = x_i + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

We integrate the equation for x from 0 to each of the $\tau_{\nu,i}$ consecutively. For each $x_i$ we find the temperature and then intensity using Gauss quadratures.

# Requirements for a formal solver

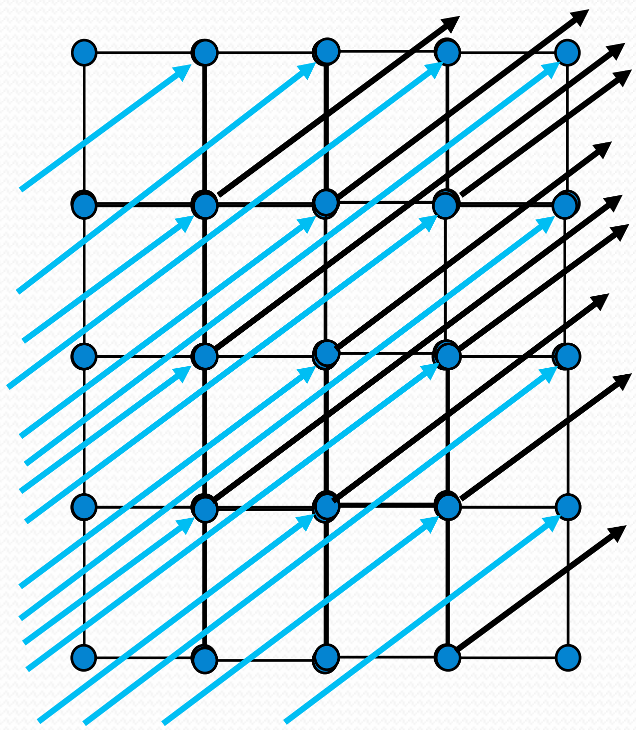
- RK is good to study the properties of your environment, selecting the grid etc.
- For practical applications the solver must be quick and stable. It should be able to achieve good accuracy on the prescribed grid.
- The formal solver should not propagate/amplify errors which may be deadly of we need iterations.
- These requirements force us to use finite differences schemes.

# Method classification

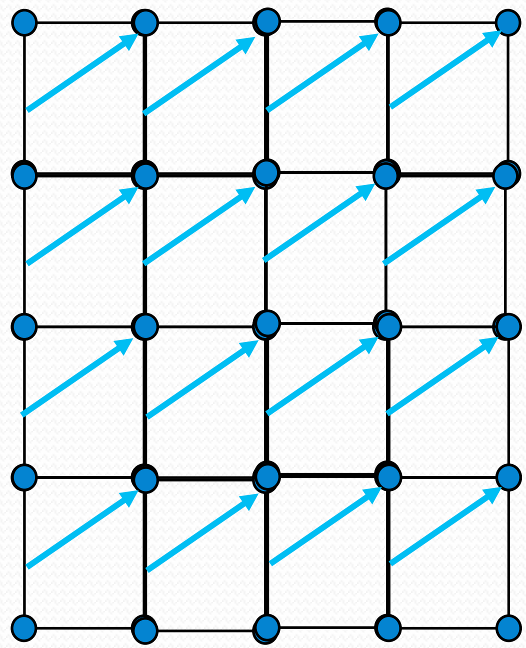
- RT is solved along rays or *characteristics* that do not necessarily coincide with the selected grid.
- Individual ray can be followed through the whole medium boundary-to-boundary or over a short part extending the length of one grid cell.
- RT solvers based on complete rays are known as *long characteristics* methods.
- RT solvers that follow radiation through a single grid cell at a time are called *short characteristics* methods.
- In 1D there is obviously no difference between short and long characteristic methods

# Method classification

Long characteristics



Short characteristics



# Comparison of long versus short

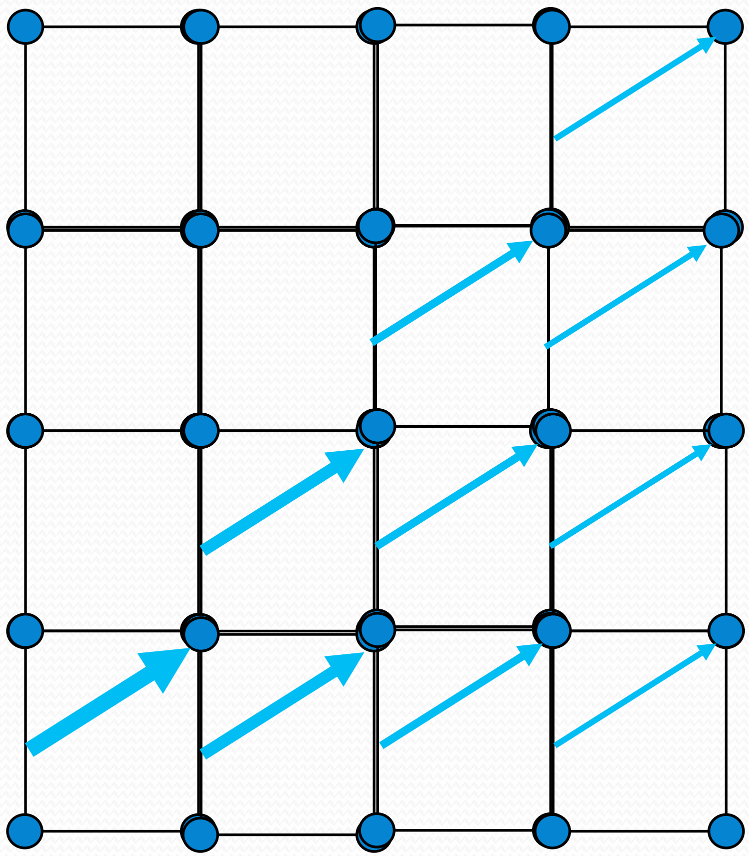
Long:

- Use both boundary conditions
- Get intensities in both directions
- Mean of the two intensities is actually a component of J
- Expansive in 2D or 3D if the geometrical grid does match the ray directions

Short:

- Fast
- Follow the geometrical grid no matter what
- Need two-directional integration to evaluate J
- Suffers from numerical “light defocussing”

# Light defocussing



# Feautrier RT solver

- Equation of radiative transfer (again)

$$\frac{dI_\nu}{dx} = -\alpha_\nu \cdot (I_\nu - S_\nu)$$

where $x$ is a geometrical distance along the ray

- Let's split the intensity in two flows: $I^+$ in the direction of increasing $x$ and $I^-$ in the opposite direction. The RT equation can be written for each direction:

$$\frac{dI_\nu^+}{dx} = -\alpha_\nu \cdot (I_\nu^+ - S_\nu)$$

$$-\frac{dI_\nu^-}{dx} = -\alpha_\nu \cdot (I_\nu^- - S_\nu)$$

We define two new variables $U = \frac{1}{2}(I^{+} + I^{-})$ and $V = \frac{1}{2}(I^{+} - I^{-})$. Now we can add/subtract the two equations of RT and divide the results by 2:

$$\frac{dI_\nu^+}{dx} = -\alpha_\nu \cdot (I_\nu^+ - S_\nu)$$

$$+$$

$$\frac{dI_\nu^-}{dx} = \alpha_\nu \cdot (I_\nu^- - S_\nu)$$

$$\frac{dU_\nu}{dx} = -\alpha_\nu \cdot V_\nu$$

$$\frac{dI_\nu^+}{dx} = -\alpha_\nu \cdot (I_\nu^+ - S_\nu)$$

$$-$$

$$\frac{dI_\nu^-}{dx} = \alpha_\nu \cdot (I_\nu^- - S_\nu)$$

$$\frac{dV_\nu}{dx} = -\alpha_\nu \cdot (U_\nu - S_\nu)$$

# $2^{nd}$ order form of RT

- We substitute the derivative of $V$ in the $2^{nd}$ equation using the expression for $V$ from the $1^{st}$ equation:

$$V_\nu = -(\alpha_\nu)^{-1}\frac{dU_\nu}{dx}$$

$$\frac{dV_\nu}{dx} = -\alpha_\nu \cdot (U_\nu - S_\nu)$$

- The equations for $U$ and $V$ can be combined into a single 2nd order ODE:

$$\frac{d}{dx}\left[(\alpha_\nu)^{-1}\frac{dU_\nu}{dx}\right] = \alpha_\nu \cdot (U_\nu - S_\nu)$$

# Boundary Conditions

Boundary conditions are set in the two ends of the medium. For the smallest *x* we can write:

$$\left(\alpha_\nu\right)^{-1} \frac{dU_\nu}{dx}\bigg|_A = -V_\nu = -\frac{1}{2}\left(I_\nu^+ - I_\nu^-\right) =$$

$$= \frac{1}{2}\left(I_\nu^+ + I_\nu^-\right) - I_\nu^+ = U_\nu - I_\nu^{A+}$$

For the opposite end we have:

$$\left(\alpha_\nu\right)^{-1} \frac{dU_\nu}{dx}\bigg|_B = -V_\nu = -\frac{1}{2}\left(I_\nu^+ - I_\nu^-\right) =$$

$$= -\frac{1}{2}\left(I_\nu^+ + I_\nu^-\right) + I_\nu^- = -U_\nu + I_\nu^{B-}$$

Finite differences equation have familiar form (*note the sign in the definitions of* $a_i$ *and* $c_i$):

$$-a_i U_{i-1} + b_i U_i - c_i U_{i+1} = d_i \quad \text{for} \quad i = 2, \ldots, N-1$$

$$a_i = \frac{1}{x_{i+1} - x_{i-1}} \cdot \frac{\left(\alpha_{\nu,i}\right)^{-1} + \left(\alpha_{\nu,i-1}\right)^{-1}}{x_i - x_{i-1}}$$

$$c_i = \frac{1}{x_{i+1} - x_{i-1}} \cdot \frac{\left(\alpha_{\nu,i+1}\right)^{-1} + \left(\alpha_{\nu,i}\right)^{-1}}{x_{i+1} - x_i}$$

$$b_i = a_i + c_i + \alpha_{\nu,i}$$

$$d_i = \alpha_{\nu,i} \cdot S_{\nu,i}$$

For $i=1$ we can write a linear boundary condition:

$$U_{1\frac{1}{2}} = \frac{U_2 + U_1}{2} \approx U_1 + \frac{(\tau_2 - \tau_1)}{2} \cdot \left.\frac{dU}{d\tau}\right|_1 =$$

$$= U_1 + \frac{(\tau_2 - \tau_1)}{2} \cdot (U_1 - I^A)$$

$$\frac{(\tau_2 - \tau_1)}{2} \approx \frac{(\alpha_2 + \alpha_1)}{2} \cdot \frac{x_2 - x_1}{2}$$

$$U_1 \cdot \underbrace{\left[1 + \frac{(\alpha_2 + \alpha_1)}{2} \cdot (x_2 - x_1)\right]}_{b_1} - \underbrace{1}_{c_1} \cdot U_2 =$$

$$= \underbrace{\frac{(\alpha_2 + \alpha_1)}{2} \cdot (x_2 - x_1)}_{d_1} \cdot I^A$$

… or we can write quadratic boundary condition:

$$\frac{U_1+U_2}{2}=U_1+\delta\tau\frac{dU}{d\tau}\bigg|_{\tau_1}+\frac{\delta\tau^2}{2}\frac{d^2U}{d\tau^2}\bigg|_{\tau_1}+\dots$$

$$\delta\tau\approx\frac{(\alpha_2+\alpha_1)}{2}\cdot\frac{(x_2-x_1)}{2}$$

$$U_2\approx U_1+2\delta\tau\cdot(U_1-I_\nu^A)+\delta\tau^2\cdot(U_1-S_1)$$

$$U_1\cdot\underbrace{\left[1+2\delta\tau+\delta\tau^2\right]}_{b_1}-\underbrace{1}_{c_1}\cdot U_2=$$

$$=\underbrace{2\delta\tau\cdot I^A+\delta\tau^2 S_1}_{d_1}$$

The case of *i*=*N* is similar

For semi-infinite medium boundary condition at ∞ looks a bit different:

$$\begin{cases} \dfrac{dU_\nu}{d\tau} = U_\nu - I_\nu^+ \\ \begin{cases} I_\nu^+\big|_\tau \approx \int_\tau^\infty S_\nu(t) e^{-(t-\tau)} dt \\ S(t) = S(\tau) + (t-\tau)\dfrac{dS}{dt}\Big|_{t=\tau} \end{cases} \end{cases} \Rightarrow I_\nu^+\big|_{\text{deep}} = S_\nu + \frac{d\,S_\nu}{d\tau} \Rightarrow U_\nu - \frac{dU_\nu}{d\tau} = S_\nu + \frac{d\,S_\nu}{d\tau}$$

$$c_N = \frac{1}{\delta\tau} - \frac{1}{2}$$

$$b_N = \frac{1}{\delta\tau} + \frac{1}{2}$$

$$d_N = \tfrac{1}{2}(S_{\nu,N-1} + S_{\nu,N}) + \frac{(S_{\nu,N-1} - S_{\nu,N})}{\delta\tau}$$

This is known as diffusion boundary condition

# How does this work?

- Select direction.
- Setup a 1D grid and compute opacities, sources function and optical steps.
- Compute the Feautrier coefficients including those given by the boundary conditions.
- Solve 3-diagonal SLE
- As free lunch you get the contribution of this ray to the angle-averaged intensity J which is U and to the flux divergence which is V.

# Hermitian method

Taylor expansion for the intensity in point $\tau_i$:

$$I_{i+1} = I_i + \sum_{n=1}^{4} \frac{\delta_i^n}{n!} \frac{d^n I}{d\tau^n}$$

$$I' = \frac{dI}{d\tau} + \delta_i \frac{d^2 I}{d\tau^2} + \frac{1}{2}\delta_i^2 \frac{d^3 I}{d\tau^3} + \frac{1}{6}\delta_i^3 \frac{d^4 I}{d\tau^4} \quad \Bigg| \times(-\delta_i/2)$$

$$I'' = \frac{d^2 I}{d\tau^2} + \delta_i \frac{d^3 I}{d\tau^3} + \frac{1}{2}\delta_i^2 \frac{d^4 I}{d\tau^4} \quad \Bigg| \times \delta_i^2/12$$

$$I_{i+1} = I_i + \tfrac{\delta_i}{2}\left(I_i' + I_{i+1}'\right) + \tfrac{\delta_i^2}{12}\left(I_i'' - I_{i+1}''\right)$$

$$I' = \alpha \cdot \left(I - S\right)$$

$$I'' = \alpha \cdot \left[\alpha \cdot \left(I - S\right) - S'\right] + \alpha' \cdot \left(I - S\right)$$

$$I_{i+1} = q_i \cdot I_i + p_i$$

# Attenuation operator solver

- Solution of RT over one grid cell can be written:

$$I_\nu(\tau_{i+1}) = e^{-(\tau_{i+1}-\tau_i)} \cdot I_\nu(\tau_i) +$$

$$+\int_{\tau_i}^{\tau_{i+1}} S_\nu(t) \cdot e^{-(\tau_{i+1}-t)} dt$$

where τ is the optical path along the ray

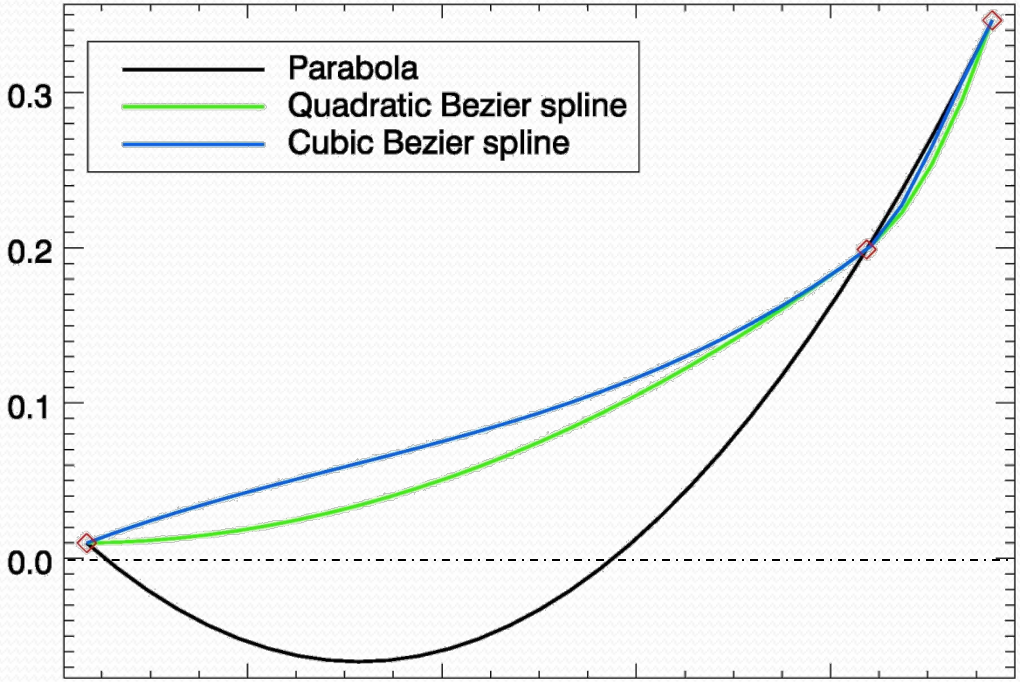
- Suppose S slowly changes with τ which can be approximated by a linear function. Then we can take the integral analytically!

$$S_\nu(\tau) = \left[ \frac{(\tau_{i+1}-\tau)}{(\tau_{i+1}-\tau_i)} S_{\nu,i} + \frac{(\tau-\tau_i)}{(\tau_{i+1}-\tau_i)} S_{\nu,i+1} \right]$$

$$I_\nu(\tau_{i+1}) = I_\nu(\tau_i) \cdot e^{-(\tau_{i+1}-\tau_i)} + \eta_{\nu,i}$$

# Source Function Approximation

Quadratic approximation for the source function is better than linear, but …

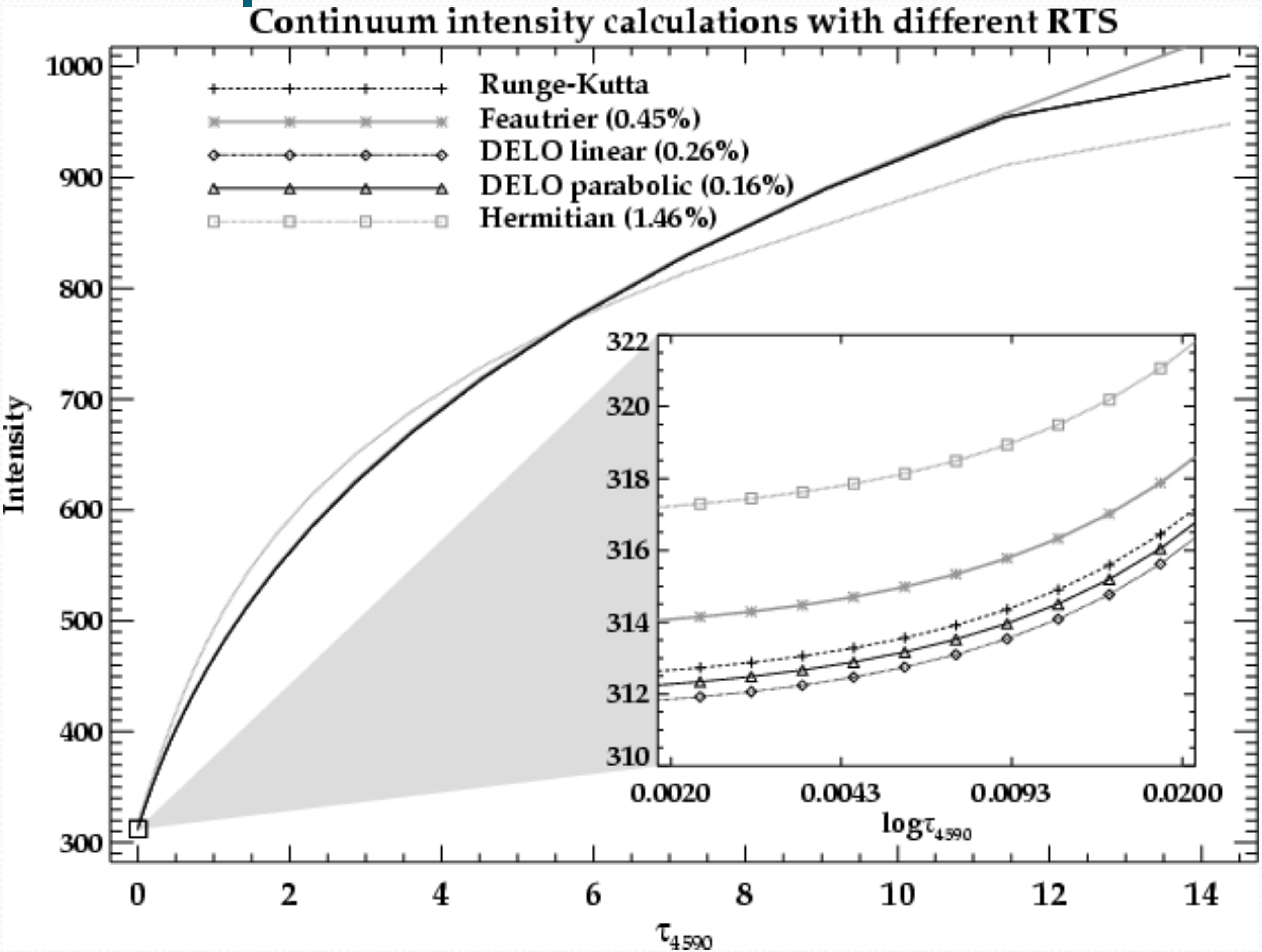



**Bezier splines are a much more robust alternative**

# How does this work?

- Short characteristics result in recurrence relation between $I_i$ and $I_{i+1}$
- Select direction
- For starting grid points incoming intensity is given by boundary conditions
- Compute the opacity and the source function in the grid points and interpolate for the up-stream and down-stream points (for quadratic schemes).
- Compute intensities for all points in the next layer.

# Comparison of the solvers



# Home work 4

Compute spectral synthesis using a method of your choice for a static 1D model atmosphere of the Sun. For a fixed geometrical depth grid and wavelength grid you are given a 2D array of opacities and 2D array of source function. The boundary conditions: no radiation enters through the surface and the flux spectrum at the deepest atmosphere point is given.