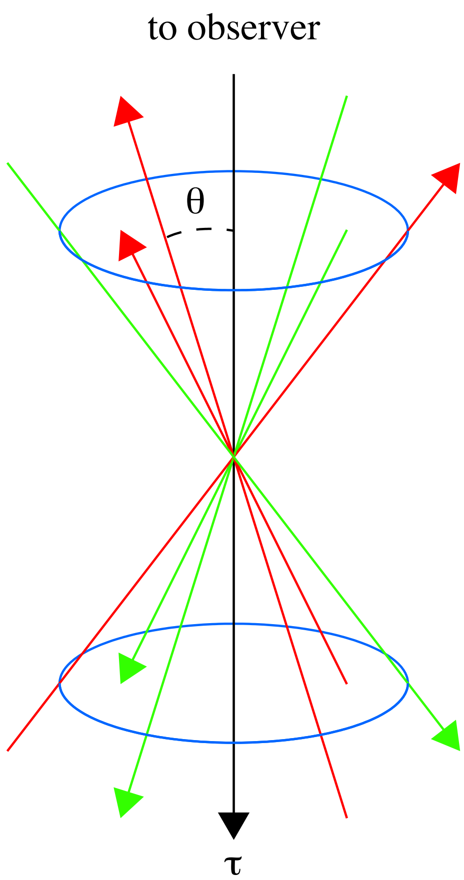
Physics of planetary atmospheres

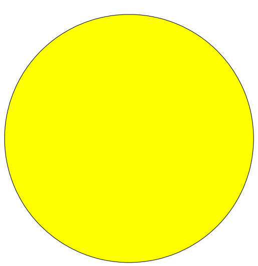
Lecture 3:
Introduction to
Radiative Energy Transport

Part II

Coordinate system

- It is convenient to exploit symmetries in the setup for handling the angles:
- What works for incoming radiation but may not work so well for describing phase variations:





Two-streams concept

- When describing radiative energy transport in planetary atmosphere we can intuitively identify two flows: “incoming” or “down” and “outgoing” or “up”.
- Small thickness of the atmosphere relative to the radius helps simplifying it even more.
- In this case for any point in the planet the external source (star) is at some zenith angle θ while outgoing radiation is distributed over 2π solid angle.

Moments of intensity

- Total intensity J:

$$J = \int_0^{2\pi} d\phi \int_{-1}^{+1} I d\mu$$

$$J \uparrow = \int_0^{2\pi} d\phi \int_0^{+1} I d\mu, \quad J \downarrow = \int_0^{2\pi} d\phi \int_{-1}^{0} I d\mu$$

- Flux:

$$F \uparrow = \int_0^{2\pi} d\phi \int_0^{+1} I \mu d\mu, \quad F \downarrow = \int_0^{2\pi} d\phi \int_{-1}^{0} I \mu d\mu$$

- Higher moments are occasionally used for RT.

Writing RT equations for two fluxes

- Let's introduce two new variables,
  total flux: $F_+ = F\uparrow + F\downarrow$
  and net flux: $F_- = F\uparrow - F\downarrow$
- Now we are going to reformulate the RT equation for these variables.
- Let's start with RT equation for intensity with isotropic scattering. The scattering part of the source function is simply set by J/4π.

$$\mu \frac{dI_\lambda}{d\tau} = I_\lambda - \omega_0 \cdot \frac{J_\lambda}{4\pi} - (1 - \omega_0) \cdot B_\lambda$$

Deriving RT equations for up a down fluxes

- Let's integrated the RT equation over μ and ϕ:

$$\frac{dF_\uparrow}{d\tau} = \quad J_\uparrow - \omega_0 \cdot \frac{J}{2} - 2\pi(1-\omega_0)\cdot B_\lambda$$

$$\frac{dF_\downarrow}{d\tau} = -J_\downarrow + \omega_0 \cdot \frac{J}{2} + 2\pi(1-\omega_0)\cdot B_\lambda$$

- We have two equations and many unknown functions even though they are not independent.
- Additional constraints are needed.

Eddington coefficients

- Definition:

$$\epsilon_\uparrow \equiv \frac{F_\uparrow}{J_\uparrow}, \quad \epsilon \equiv \frac{F_+}{J}$$

- The big leap is to assume that these coefficients are constant in τ.
- There is no reason why it should be true but one should expect correlation between flux and mean intensity.

RT with constant Eddington coefficients

- The transformation for the up/down fluxes:

$$\frac{dF_\uparrow}{d\tau} = \quad J_\uparrow - \omega_0 \cdot \frac{J}{2} - 2\pi(1-\omega_0)\cdot B_\lambda$$

$$\frac{dF_\downarrow}{d\tau} = -J_\downarrow + \omega_0 \cdot \frac{J}{2} + 2\pi(1-\omega_0)\cdot B_\lambda$$

$$\frac{dF_\uparrow}{d\tau} = \quad F_\uparrow \cdot \left(\frac{1}{\epsilon_\uparrow} - \frac{\omega_0}{2\epsilon}\right) - \frac{\omega_0 F_\downarrow}{2\epsilon} - 2\pi(1-\omega_0)\cdot B_\lambda$$

$$\frac{dF_\downarrow}{d\tau} = -F_\downarrow \cdot \left(\frac{1}{\epsilon_\downarrow} - \frac{\omega_0}{2\epsilon}\right) + \frac{\omega_0 F_\uparrow}{2\epsilon} + 2\pi(1-\omega_0)\cdot B_\lambda$$

- Now we have two equations and two unknown functions.

Further assumptions

- If the ratios of flux to mean intensity for up and down directions are constant, there is no reason why they should be different (same rays, same local conditions).
- Then we can denote $\epsilon' \equiv \epsilon_{\uparrow} = \epsilon_{\downarrow}$ and simplify our equations:

$$\frac{dF_{\uparrow}}{d\tau} = \quad F_{\uparrow} \cdot \gamma_a - F_{\downarrow} \cdot \gamma_s - B_{\lambda} \cdot \gamma_B$$

$$\frac{dF_{\downarrow}}{d\tau} = -F_{\downarrow} \cdot \gamma_a + F_{\uparrow} \cdot \gamma_s + B_{\lambda} \cdot \gamma_B$$

Note, that equations are coupled by scattering!

Stellar analogy

- RT equations:

$$\mu \frac{dI_\uparrow}{d\tau} = \quad I_\uparrow - \omega_0 \cdot \frac{J}{4\pi} - (1-\omega_0) \cdot B_\lambda$$

$$\mu \frac{dI_\downarrow}{d\tau} = -I_\downarrow + \omega_0 \cdot \frac{J}{4\pi} + (1-\omega_0) \cdot B_\lambda$$

$$+ : \mu \frac{dI_+}{d\tau} = I_-$$

$$- : \mu \frac{dI_-}{d\tau} = I_+ - \omega_0 \cdot \frac{J}{2\pi} - 2(1-\omega_0) \cdot B_\lambda$$

- Feautrier solution:

$$\mu^2 \frac{d^2 I_+}{d\tau^2} = I_+ - \omega_0 \cdot \frac{J}{2\pi} - 2(1-\omega_0) \cdot B_\lambda$$

Feautrier method

- It is stable and fast
- It requires two boundary conditions and we do have them: deep inside a star I_+ is given by Planck function. At the surface I_+ is equal to I.
- No additional closure assumptions are needed.
- J is I_+ integrated over μ. Solving this equation for many directions is nearly as easy as for the single direction.
- Discretisation leads to a 3-diagonal system of linear equations.

Pure scattering

- Equilibrium means the net energy flux through any layer is constant:

$$F_- = F_\uparrow - F_\downarrow, \quad F_+ = F_\uparrow + F_\downarrow$$

$$\int \frac{dF_-}{d\tau} d\lambda = 0 = \int (\gamma_a - \gamma_s) F_+ d\lambda - 2 \int B_\lambda \cdot \gamma_B d\lambda$$

- In pure scattering regime we get:

$$\omega_0 = 1, \gamma_B = 0 \Rightarrow \gamma_a \equiv \left(\frac{1}{\epsilon'} - \frac{\omega_0}{2\epsilon} \right) = \frac{\omega_0}{2\epsilon} \equiv \gamma_s$$

$\Rightarrow \epsilon = \epsilon'$ No absorption/emission, no heating or cooling of gas!

Pure absorption

$$\omega_0 = 0 \Rightarrow \gamma_a = \frac{1}{\epsilon'}, \gamma_s = 0 \text{ and } \gamma_B = 2\pi$$

$$\begin{aligned} \frac{dF_\uparrow}{d\tau} &= \quad F_\uparrow \cdot \gamma_a - B_\lambda \cdot \gamma_B \\ \frac{dF_\downarrow}{d\tau} &= -F_\downarrow \cdot \gamma_a + B_\lambda \cdot \gamma_B \end{aligned}$$

Let’s see if we can estimate the value of ε’ in a very simplified case. Let’s take isothermal medium. Then B is constant in τ.

$$F_{\uparrow 1} = F_{\uparrow 2} e^{-\gamma_a \Delta\tau} + \frac{\gamma_B}{\gamma_a} B \left(1 - e^{-\gamma_a \Delta\tau}\right)$$

$$F_{\downarrow 2} = F_{\downarrow 1} e^{-\gamma_a \Delta\tau} + \frac{\gamma_B}{\gamma_a} B \left(1 - e^{-\gamma_a \Delta\tau}\right)$$

Pure absorption in optically thick medium

- Further simplification is achieved assuming the path between points 1 and 2 to be optically thick:

$$F_{\uparrow 1} = \frac{\gamma_B}{\gamma_a} B = 2\pi\epsilon' B$$

$$F_{\downarrow 2} = \frac{\gamma_B}{\gamma_a} B = 2\pi\epsilon' B$$

- From the moment equations we know that in isothermal, absorbing, optically thick case flux is equal to ½ of total intensity. This means that ε=ε‘=½.

Transmission function

- Substituting expression ε to γ 's we finally get:

$$\gamma_a = 2 - \omega_0, \gamma_s = \omega_0 \text{ and } \gamma_B = 2\pi(1 - \omega_0)$$

- Transmission function was defined in lecture 2 assuming isothermal medium:

$$F_{\uparrow 1} = F_{\uparrow 2} T + \pi B(1 - T)$$
$$F_{\downarrow 2} = F_{\downarrow 1} T + \pi B(1 - T)$$

Transmission function

- γ_a and therefore ε appear in the coefficient in front of optical depth in the formal solution:

$$F_{\uparrow 1} = F_{\uparrow 2} e^{-\gamma_a \Delta\tau} + \frac{\gamma_B}{\gamma_a} B \left(1 - e^{-\gamma_a \Delta\tau}\right)$$

$$F_{\downarrow 2} = F_{\downarrow 1} e^{-\gamma_a \Delta\tau} + \frac{\gamma_B}{\gamma_a} B \left(1 - e^{-\gamma_a \Delta\tau}\right)$$

- Comparing the two sets of equations we get more realistic expression for transmission T:

$$T = e^{-\gamma_a \Delta\tau} = e^{-\Delta\tau/\epsilon}$$

Two-stream approximation solution (isotropic scattering)

- We start with expression for up and down fluxes:

$$\frac{dF_\uparrow}{d\tau} = \quad F_\uparrow \cdot \gamma_a - F_\downarrow \cdot \gamma_s - B_\lambda \cdot \gamma_B$$

$$\frac{dF_\downarrow}{d\tau} = -F_\downarrow \cdot \gamma_a + F_\uparrow \cdot \gamma_s + B_\lambda \cdot \gamma_B$$

- Adding and subtracting these equations gives:

$$\frac{dF_+}{d\tau} = (\gamma_a + \gamma_s) \cdot F_-$$

$$\frac{dF_-}{d\tau} = (\gamma_a - \gamma_s) \cdot F_+ - 2\gamma_B \cdot B$$

Two-stream approximation solution

- Substituting the expression for the net flux to the derivative in 2nd equation we get:

$$\frac{d^2 F_+}{d\tau^2} = D \cdot F_+ - 2\gamma_B(\gamma_a + \gamma_s) \cdot B$$

$$\text{Diffusivity factor } D \equiv \sqrt{(\gamma_a + \gamma_s)(\gamma_a - \gamma_s)}$$

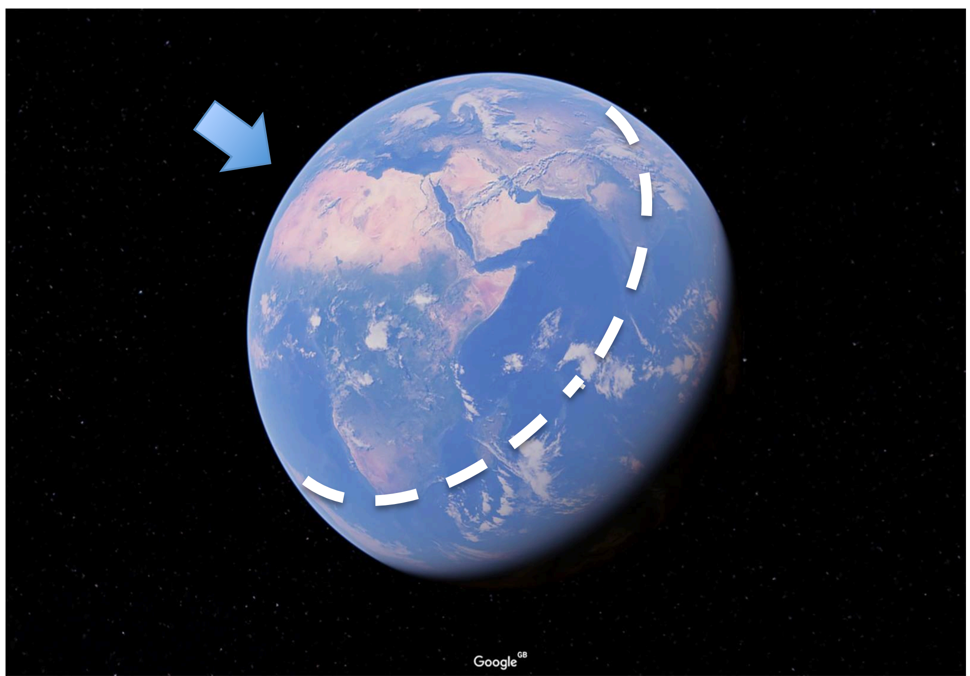
- The big assumption here is that γ's are not functions of τ.
- Thus an atmosphere must be divided in layers, thin enough for this to be true.

Steps to realism

- First step: divide the atmosphere in many isothermal layers. In this case we can use the solutions for the two streams directly assuming the value of temperature.
- Second step: Use linear approximation for B in each layer $B(\tau)=B_1+(\tau-\tau_1)(B_2-B_1)/(\tau_2-\tau_1)$. That adds small complication but the integral of B can still be calculated analytically.
- Third step: non-isotropic scattering.

Non-isotropic scattering

- Of course, scattering in optically thin part of planetary atmospheres is non-isotropic.
- The scattering pattern is expected to be symmetric relative to the irradiation direction:



Single scattering geometry

- Let's define two directions: one to the star θ and one to the observer θ'.
- The cos(θ-θ')=μ'' is known from the spherical geometry course that nobody takes any longer:

$$\mu'' = \mu\mu' + \sqrt{(1-\mu^2)(1-\mu'^2)}\cos(\phi-\phi')$$

- If the phase function P depends only on θ-θ' then it must be invariant for transposition of the image.

Constructing scattering phase function

- Another important property of P is that it should be invariant of coordinate system transformation.
- Instead of deriving an analytical form of P we can construct it out of a family of spherical functions with the right properties or by selecting an analytical expression with fitting parameters (Henyey-Greenstein).

Deep inside

- In the optically deep part of the atmosphere scattering becomes equivalent to absorption because photons travel very short distance staying in equilibrium with the gas.
- Thus J=B and S=B.
- Flux is then defined by the gradient of B − diffusion approximation.

Real thing

- Using flux concept kills our ability of treating 3D atmospheres,
- We totally ignored convection and horizontal energy transport by winds, surface gradient of temperature and albedo.
- In real climate model RT is solved for intensity along many rays but the two-stream or Feautrier method are frequently used.