Stellar Atmospheres, Autumn 2007

Lecture 3. Opacities

From Lecture 2, notes

Flux in terms of intensity
\[ F_\nu = \int I_\nu \cos \theta \, d\omega \]  (linear in \( \cos \theta \equiv \mu \))

K integral
\[ K_\nu = \frac{1}{4\pi} \int I_\nu \cos^2 \theta \, d\omega \]  (quadratic in \( \cos \theta \))

Optical depth
\[ \tau_\nu = \int_0^L \kappa_\nu \rho \, ds \]  [unitless]

Source function
\[ S_\nu = \frac{i_\nu}{\kappa_\nu} = \frac{\text{emission coefficient}}{\text{mass absorption coefficient}} \]
\([S_\nu] = [I_\nu] \]

Example:
\[ S_\nu = B_\nu \] (Planck function)

Assuming LTE
\[ d\tau_\nu = \kappa_\nu \rho \, ds \]

"Mass extinction coefficient" or "mass absorption coefficient"
Units: cm\(^2\)/g or m\(^2\)/kg
Q: *What is the difference between \( \kappa_\nu \) and \( \kappa_\lambda \)?*

One may form a mean of \( \kappa \) over frequency or wavelength, e.g. a harmonic mean with \( dB_\nu(T)/dT \) as weighting function:

\[
\frac{1}{\kappa_R} = \frac{\int \frac{1}{\kappa_\nu} dB_\nu(T)/dT \, dv}{\int dB_\nu(T)/dT \, dv}
\]

For optically thick gas (like in stellar interiors) one may find for the total radiative flux

\[
\mathcal{F} \propto \frac{1}{\kappa_R} \frac{dT}{ds}
\]

Obviously, this is analogous to the Equation of Heat Transfer with \( 1/\kappa_R \) playing a role corresponding to conductivity. \( \kappa_R \), and similar means, are often called ”opacity”. Sometimes the term is also used for \( \kappa_\nu \).
The Einstein coefficients (Einstein 1917)

Probability that atom will emit a quantum $h\nu$ in time $dt$ and solid angle $d\omega$ is $A_{ul} dt d\omega$

Contribution of spontaneous emission coeff. $j_v$ is $j_v \rho = N_u A_{ul} h\nu$ where $N_u$ is the number of atoms excited to level $u$ per unit volume.

Probability for stimulated emission per atom giving quantum $h\nu$ in time $dt$ and solid angle $d\omega$ is $B_{ul} I_v dt d\omega$

Probability for absorption by atom of quantum $h\nu$ in time $dt$ and solid angle $d\omega$ is $B_{ul} I_v dt d\omega$

Energy of radiation absorbed per unit path length and frequency, from beam with intensity $I_v$:

$\kappa_v \rho I_v = N_i B_{ul} I_v h\nu - N_i B_{ul} I_v h\nu$
The Einstein coefficients (Einstein 1917)

Probability that an atom will emit a quantum $h\nu$ in time $dt$ and solid angle $d\omega$ is $A_{\alpha\ell} dt \, d\omega$

Contribution of spontaneous emission coeff. $j_{\nu}$ is $j_{\nu} \rho = N_u A_{u\ell} \, h\nu$ where $N_u$ is the number of atoms excited to level $\alpha$ per unit volume.

Probability for stimulated emission per atom giving quantum $h\nu$ in time $dt$ and solid angle $d\omega$ is $B_{u\ell} j_{\nu} \, dt \, d\omega$

Probability for absorption by atom of quantum $h\nu$ in time $dt$ and solid angle $d\omega$ is $B_{u\ell} j_{\nu} \, dt \, d\omega$

Energy of radiation absorbed per unit path length and frequency, from beam with intensity $I_{\nu}$:

$$\kappa_{\nu} \rho \, I_{\nu} = N_u B_{u\ell} I_{\nu} \, h\nu - N_{\ell} B_{u\ell} I_{\nu} \, h\nu$$
Now, assume Thermodynamical Equilibrium!

\[ \frac{N_d}{N_l} = \frac{g_d}{g_l} \exp(-\hbar \nu / kT) \]

Boltzmann distribution, \( g_u \) and \( g_l \) statistical weights

Also, number of downward transitions = number of upward transitions per time unit. Collision transitions up and down cancel in TE => \( N_u A_{ul} + N_u B_{ul} I_\nu = N_l B_{lu} I_\nu \)

Solve for \( I_\nu \) and use the Boltzmann distribution:

\[ I_\nu = \frac{A_{ul}}{(g_l / g_u) B_{lu} \exp(\hbar \nu / kT) - B_{ul}} \]

cf Planck’s law:

\[ I_\nu = \frac{2\hbar \nu^3}{c^2} \frac{1}{\exp(\hbar \nu / kT) - 1} \]

This obviously suggests

\[ \begin{cases} B_{ul} = (g_l / g_u) B_{lu} \\ A_{ul} = 2\hbar \nu^3 / c^2 B_{ul} \end{cases} \]

Note: Planck’s law can actually be proven this way, if one uses the low energy limit (Rayleigh–Jean’s law), see Gray, page 115
Now, assume Thermodynamical Equilibrium!

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Boltzmann distribution, \( g_u \) and \( g_l \) statistical weights

Also, number of downward transitions = number of upward transitions per time unit. Collision transitions up and down cancel in TE \( \Rightarrow \) \( N_u A_{ul} + N_u B_{ul} I_\nu = N_l B_{lu} I_\nu \)

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\[ \begin{align*}
B_{ul} &= (g_l/g_u) B_{lu} \\
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\end{align*} \]

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BUT, these are only atomic quantities -- independent of assumption of TE. Thus, the relations are generally valid!
Microscopic absorption mechanisms

bound-free (bf) absorption -- fotoionization

free-free (ff) absorption -- cf Brehmsstrahlung

bound-bound (bb) absorption - line absorption

scattering
The determination of cross sections
(like $B_{ij}$ or $B_{ik}$)

- Many measurements exit, not the least of $A_{ji}$ (life times). See Vald data base (Piskunov et al.). Difficult for excited states. Often missing for photo-ionization.

- Calculations:
  a) Classical electrodynamics calc. Gives dimensionally correct expressions, but may be orders of magnitude wrong, not the least for weaker transitions
  b) Semiclassical (atom QM, electromagnetic field Classical). Problem to determine wave function describing many-electron systems. Hartree-Fock self-consistent field method (sum over electron pairs $\rightarrow$ spherical average for each electron. Wave function written as Slater determinant. In the end the line strength

$$S = a_0^2 e^2 \sigma^2 S(M)S(L), \quad \sigma^2 \sim \{ \int P_n l P_{nl} r \, dr \}^2$$

  c) Full QM $\uparrow$ Radial part

Strengths of multiplet and of line in multiplet
From micro to macro:

- Typical atomic dimensions: Bohr radius $a_0 = 5 \times 10^{-9}$ cm.
- Typical atomic cross section: $\pi a_0^2 = 8 \times 10^{-17}$ cm$^2$.
- Typical max density in stellar atmosphere gas: $N_a = 10^{17}$ atoms/cm$^3$.
- Typical distance between atoms: $1/3\sqrt{N_a} > 10^{-6}$ cm, i.e. much more than atomic dimension.
- Typical max cross section area of one cm$^3$: $8 \times 10^{-17} \times 10^{17} = 8 \times 10^{-17}$ x $10^{17} ->$ stellar atmosphere matter may be quite opaque at depth.

However, most atoms are not excited so much if not ionized -> only opaque at very short wavelengths. E.g. for hydrogen and $\lambda > 912$ Å the Boltzmann factor $\exp(-h\nu/kT) = 10^{-10.3\theta}$ with $\theta = 5040/T$ brings the cross section down very considerably.
This suggests that we can just sum up all individual contributions without taking possible collective phenomena into account:

$$\kappa_{\nu \rho} = N_{\ell} B_{\ell \alpha} h \nu$$

We may as well include the stimulated emission into this (since it also scales with $I_{\nu}$), and get

$$\kappa_{\nu \ell \rho} = N_{\ell} B_{\ell \alpha} h \nu (1-e^{-h\nu/kT})$$

We can instead take the cross section for absorption of atom at level $\ell$:

$$\alpha_{\ell} = B_{\ell \alpha} h \nu$$

and thus get

$$\kappa_{\nu \ell \rho} = N g_{\ell} \exp(-\chi_{\ell}/kT)/Q \alpha_{\ell} (1-e^{-h\nu/kT})$$

where $Q = \sum g_{i} \exp(-\chi_{i}/kT)$

Here, we have assumed the Boltzmann distribution to be valid (LTE).
Important example: Hydrogen atoms

$E_{\text{eV}}$

13.6  
10.2  
0.0

$n = \infty$

$n = 2$

$n = 1$

Lyman continuum $\lambda < 912 \text{ Å}$

$\text{Ly} \alpha, \lambda = 1216 \text{ Å}$

Wavelength - [Å]
Important example: Hydrogen atoms

\[
\begin{align*}
E (\text{eV}) & \quad n = \infty \\
13.6 & \\
10.2 & \\
0.0 & \\
\end{align*}
\]

Balmer \( \alpha \), \( \lambda = 6563 \ \text{Å} \)

Balmer continuum \( \lambda < 3646 \ \text{Å} \)
Important example: Hydrogen atoms

E (eV)

13.6
10.2

n = \infty

n = 3

Paschen continuum \( \lambda < 8207 \text{ Å} \)

Paschen \( \alpha, \lambda = 18756 \text{ Å} \)
Rydberg’s formula

\[ \frac{1}{\lambda} = R \, Z^2 \left( \frac{1}{n^2} - \frac{1}{m^2} \right) \]

- \( Z \) = atomic number
- \( m > n \)

\[ R = R_{\infty} \left\{ 1 - \frac{m_e}{m_e + M} \right\} \]

- \( R_{\infty} = 1.097 \, 373 \, 105 \, \text{cm}^{-1} \)
- \( M \) = atomic mass, \( m_e \) = electron mass

\[ E_n = - R \, h \, c \, Z^2/n^2 \]
Absorption by neutral hydrogen (H I)

bf absorption
\[ \alpha_n = 1.044 \times 10^{-26} \ g_n' \ \lambda^3 / n^5 \quad (\lambda \text{ in } \text{Å}, \text{result in cm}^2) \]

Kramers (1923) ≠ Cramers!

"Gaunt" factor correcting classical result to QM \( g_n' \approx 1 \)

**Q:** Physical interpretation of \( \lambda^3 \) dependence?

See Gray, Fig. 8.2! Note at ionization limits: \( \nu = hRc/n^2 \)

\( \Rightarrow \) \( \alpha_n \) at ionization limit \( \sim n \).

Also note that \( g_n = 2n^2 \) so that we get

\[ \kappa(\text{HI-bf}) = 1.044 \times 10^{-26} \sum_{n_0}^{\infty} [\lambda/n]^3 \ g_n' \ 10^{-\theta \chi} \]

Higher levels may be approximated by integral, see Gray p 145 ff
**HI, ff absorption**

Classically

\[ d\alpha_{ff} = \left(\frac{2}{3\sqrt{3\pi}}\right) \frac{h^2 e^2 R}{m^3 v^3} \ dv \]

\( v \) electron speed. **Q:** Why \( v \) in denominator?

Integrate over Maxwell distribution of electrons!

=>

\[ \alpha_{ff} = \text{const} \lambda^3 / \sqrt{T} \text{ per H ion and electron} \]

QM: Correct again in Gaunt factor \( g_{ff} \) and multiply with \( N_{\text{ion}} \) and \( N_e \) and divide by \( N_0 \) (= # H atoms) to get abs. coeff per H atom:

\[ \kappa(\text{Hff}) = \text{const} \lambda^3 g_{ff} T 10^{-\theta I} \]

Here, \( I \) is the ionization energy, 13.6 eV.

**Q:** Explain the linear \( T \) dependence!
The H\textsuperscript{-} absorption

A bound state, dissociation energy 0.754 eV $\leftrightarrow$ 16 450 Å.
Wildt (1939): H\textsuperscript{-} dominant absorber in solar atmosphere.

How much H\textsuperscript{-} is there?
Saha:

$N(\text{H I}) = 1.34 \ T^{5/2} \ T^{-0.754} \ \theta / P_e$

$N(\text{H-})$

How much H\textsuperscript{-} do you find in the solar atmosphere?
One finds about 2 H\textsuperscript{-} per $10^8$ H I.

Q: Why is it still important?
The H⁻ absorption, cont.

We get then
\[ \kappa(\text{H}_\text{bf}^-) = 4.158 \times 10^{-10} \alpha_{\text{bf}}(\text{H}^-) P_e \theta^{5/2} \times 10^{0.752} \theta \text{ (cm}^2 \text{ per H atom)} \]

There is also free-free H⁻, again with a \( \lambda^3 \) behaviour.

\[ \alpha_{\text{ff}}(\text{H}^-) \text{ in cm}^2 \text{ per ion and dyne electron pressure} \]
Other hydrogen contributors
• $\text{H}_2^+$ (near UV, requires H I and protons)
• $\text{H}_2^-$ (ff in IR, cool stars)
• H+H (Quasi molecule, UV).

Other negative ions
• He$^-$ (ff in IR for cool stars)
• C$^-$, N$^-$, C$_2^-$, ... (ff in IR, cool stars, note C stars)

Helium
• Important for hot stars, but more He II than He I. **Q: Why?**
• He I first series limit at 504 Å (ion. energy 24.6 eV)
• Two systems, singlet and triplet. Edges close to H.
• He II similar to H I but shifted to $\lambda/4$ (cf Rydberg’s formula), 227 Å Ly series limit.
Metals matter!

- C, Si, Al, Mg, Fe, … bf, mostly in UV
- Complicated to calculate
  *Opacity project*, quantum-defect method.
  
  Note resonances.

  absorption also included.
Scattering

• Scattering vs electrons.
  *Thomson scattering*: $\alpha_e = 6.6510^{-25}\text{ cm}^2/\text{electron}$.
    Independent of frequency! Averaged over angle, phase function for dipole scattering: $(1+\cos^2\theta)/2$.
  *Q*: cf Compton scattering!
  $N_e$ from Saha’s law. Note different electron contributors for different temperatures. Which?
  For which stars is electron scattering important?

• *Rayleigh scattering* vs H I, He I and H$_2$.
  Note $\lambda^{-4}$ dependence! For which stars are these important?
The overall picture

• Se plots distributed at lecture!