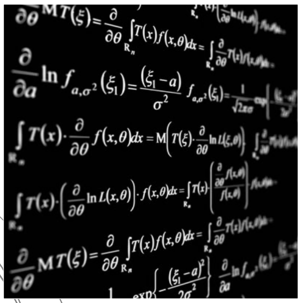
Cosmology AS7009, 2008
Lecture 3



Outline

- The Robertson-Walker metric
- Proper distance
- Is the Universe Infinite?
- Computational tools:
  - Friedmann equation
  - Fluid equation
  - Acceleration equation
  - Equation of state
- Cosmic dynamics
- The cosmological constant

Covers half of chapter 3 + chapter 4 in Ryden

Recall from last time

- Metric: *A description of the distance between two points*
- Metric in 3 spatial dimensions:

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{1-\kappa x^2/R^2} + x^2\mathrm{d}\Omega^2$$

$$\text{Flat}: \quad \kappa = 0,\ x = r$$

$$\text{Negative}: \quad \kappa = -1,\ x = R\sinh(r/R)$$

$$\text{Positive}: \quad \kappa = 1,\ x = R\sin(r/R)$$

$$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$$

Distance (squared)

Curvature radius

Metric in 4D space-time

- Minkowski metric (used in special relativity) :

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

Time   Space

Distance in 4D space-time, without *curvature*.

Geodesics

- Geodesic: The shortest path between two points
- In special/general relativity: the shortest path between two points in 4D space-time
- Photons follow the *null geodesic*, for which $\mathrm{d}s^2=0$.

$$\mathrm{d}s^2 = 0 \quad \Rightarrow$$

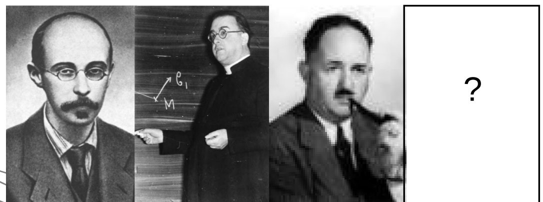
$$0 = -c^2\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2 \quad \Rightarrow$$

$$c^2\mathrm{d}t^2 = \mathrm{d}r^2 \quad \text{(for radial path)} \quad \Rightarrow$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \pm c \quad \text{(i.e. photons move with speed c)}$$

Robertson-Walker metric

(a.k.a. Friedmann-Lemaître-Robertson-Walker metric, Friedmann-Robertson-Walker or Friedmann-Lemaître metric)



Friedmann   Lemaître   Robertson   Walker

The R-W metric is an exact solution to Einstein's field equation of general relativity, assuming isotropy and homogeneity (the cosmological principle)

It is extremely important for modern cosmology!

Robertson-Walker metric II

- Valid for Universe described by general relativity (4D space-time with curvature and expanding space)
- Note: Assumes cosmological principle to hold

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2\underbrace{\left[\frac{\mathrm{d}x^2}{1-\kappa x^2/R^2} + x^2\mathrm{d}\Omega^2\right]}_{\text{Same as in 3D metric with curvature}}$$

Flat: $\kappa = 0,\ x = r$

Negative: $\kappa = -1,\ x = R\sinh(r/R)$

Positive: $\kappa = 1,\ x = R\sin(r/R)$

$\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2$

Coordinates in R-W metric

- t = proper time ('cosmic time')
- (x,θ,ϕ) or (r,θ,ϕ) = comoving coordinates

Remember:

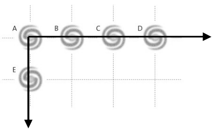
Negative curvature ($\kappa = -1$) : $x = R\sinh(r/R)$

No curvature ($\kappa = 0$) : $x = r$

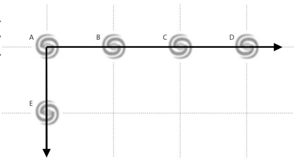
Positive curvature ($\kappa = 1$) : $x = R\sin(r/R)$

Comoving coordinates

t_1



t_2



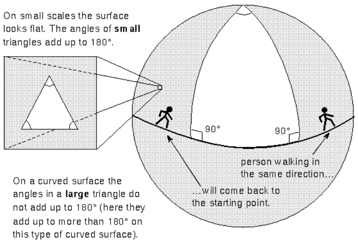
In comoving coordinates:
$d_{AB}(t_1) = d_{AB}(t_2)$

Comoving distance

- Comoving distance: A distance that stays constant despite cosmic expansion, i.e. follows the Hubble flow
- Other similar expressions: Comoving volume (often a comoving cube), comoving observers, comoving density

Is the Universe Infinite?
- Geometry -

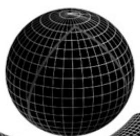
Consider a 2D world located in the surface layer of a 3D sphere

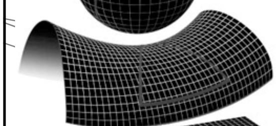


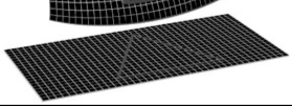
This 2D world will have finite area but no wall-like boundary

Is the Universe Infinite?
- Geometry -

Now, consider 3 spatial dimensions with curvature







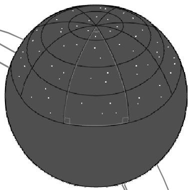
Positive curvature (κ=1) → 'closed geometry', finite Universe

Negative curvature (κ=-1) → 'open geometry', infinite Universe

Zero curvature (κ=0) → 'flat geometry', infinite Universe

Common Misconception about Modern Cosmology no. 3

- “In a closed (positively curved) universe (as illustrated in many textbooks with a picture of a 3-D sphere), we live inside a thin layer on a surface of an otherwise empty sphere. This implies that space should be more extended in certain directions (tangentially along the surface of the sphere) than in others (radially).”



No, no, no…

Is the Universe Infinite? - Topology -

- Complications:
  - Despite what is stated in many textbooks, the finiteness of space depends on *geometry* and *topology*.
  - A flat or negatively curved Universe can be finite, if the topology is non-trivial.

Asteroids:
Classic arcade game by Atari, from 1979

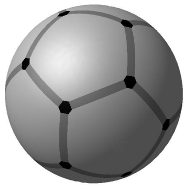
http://games.atari.com/arcade_large.php?game=asteroids

Is the Universe Infinite? - Topology -

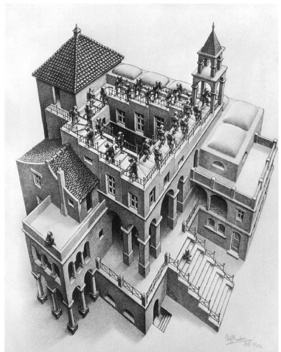
Topology of the Universe: Decribes how various parts of space are connected

Asteroids (1979): A simple of a non-trivial topology, giving a finite 2D Universe without any visible boundary (wall)

Suggestion for Literature Exercise: Topology of the Universe



Luminet's dodecahedron



Escher (1960)

Proper distance

Proper distance from observer ($r = 0$) to r:

$$d_{\mathrm{p}}(t) = a(t)\int_0^r \mathrm{d}r = a(t)r$$

- Proper distance between points A & B at time *t*: defined by spatial geodesic between A & B when scale factor is fixed at a(*t*).
- Grows over time due to cosmic expansion, i.e. *not* a comoving distance

Recession velocities I

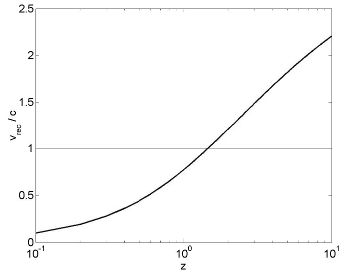
- Rate of change of the proper distance:

$$\dot{d}_{\mathrm{p}} = \dot{a}r = \frac{\dot{a}}{a}d_{\mathrm{p}}$$
$$v_p(t_0) = H_0 d_{\mathrm{p}}(t_0)$$

- This implies:

$$d_{\mathrm{p}} > \frac{c}{H_0} \rightarrow$$
$$v_p(t_0) > c$$

Recession velocities II



- v>c(!) at large distances (high z)
- But note: Not objects moving through space, but recession velocity due to expansion of space itself

Cosmic dynamics: Four important equations

- Friedmann equation
- Fluid equation
- Acceleration equation
- Equation of state

Learning how to use these is one of the major goals of this course!

Cosmic dynamics: Four important equations

What do you need them for?

- Calculating:
  ρ(t), a(t), a'(t), a''(t), T(t), t(z), D(z)
- Examples of applications:
  - Predicting fate of the Universe
  - Testing ages of astronomical objects (stars/galaxies) against cosmological models

The Friedmann equation

$$H \equiv \frac{\dot{a}}{a} \quad \Rightarrow$$

$$H(t)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2}$$

At the current time:

$$H_0{}^2 = \frac{8\pi G}{3c^2}\varepsilon_0 - \frac{\kappa c^2}{R_0^2}$$

Critical density

Critical density ≡ The energy density required to make the Universe spatially flat

$$\kappa = 0 \quad \Rightarrow$$

$$\varepsilon_{\mathrm{c}}(t) = \frac{3c^2}{8\pi G}H(t)^2$$

and

$$\varepsilon_{\mathrm{c},0} = \frac{3c^2}{8\pi G}H_0{}^2$$

Numerically, the critical energy density at the current time corresponds to 1 hydrogen atom per 200 liters, or 140 Msolar per kpc^3

The Density Parameter

Dimensionless density parameter

$$\Omega(t) \equiv \frac{\varepsilon(t)}{\varepsilon_{\mathrm{c}}(t)}$$

Rewriting the Friedmann equation:

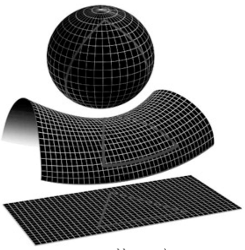
$$1 - \Omega(t) = -\frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2}$$

or, at the current time:

$$1 - \Omega_0 = -\frac{\kappa c^2}{R_0^2 H_0{}^2}$$

Once more: Energy determines geometry (curvature)

Ω and Geometry



Energy density	Curvature (geometry)
$\Omega_o>1$	κ=+1 Closed
$\Omega_o<1$	κ=-1 Open
$\Omega_o=1$	κ=0 Flat

Why only 3 possible values for κ?

The curvature κ always appears together with R_0. You can always rescale R_0 to get κ=0,+1 or -1

$$1-\Omega_0 = -\frac{\kappa c^2}{R_0^2 H_0^2}$$

The Fluid Equation

The Friedmann equation does not, by itself, allow a prediction of how the scale factor evolves with time. You also need the *fluid equation*. Assumption: The energy components of the Universe can be treated as perfect fluids on large scales.

First law of Thermodynamics:

$$\mathrm{d}Q = \mathrm{d}E + P\mathrm{d}V$$

Heat flow into or out of region → $\mathrm{d}Q$

Change in internal energy of region → $\mathrm{d}E$

Change in volume of region → $P\mathrm{d}V$

The Fluid Equation

Cosmic expansion is an adiabatic process, giving no increase in entropy → d*Q* = 0

Hence,

$$\mathrm{d}E + P\mathrm{d}V = 0$$

From this, you can derive the fluid equation:

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

The Acceleration Equation

The Friedmann equation and the fluid equation can be combined to form the acceleration equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$$

Important implication: Positive energy and pressure makes cosmic expansion slow down. But a component with negative pressure ('tension') could cause the expansion to speed up.

The Equation of State

The equation of state relates the pressure and energy of the cosmic fluids:

$$P = w\varepsilon$$

Equations of states for the some of the most common components considered in cosmology:

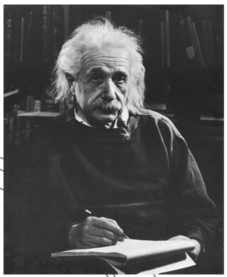
$w = 0$ (non-relativistic matter)

$w = 1/3$ (radiation and relativistic matter)

$w = -1$ (cosmological constant)

$w < -1/3$ (dark energy) → Positive acceleration

The Cosmological Constant I



In 1917 Einstein introduced a constant Λ in his equations to produce a static matter-filled Universe.

After Hubble's discovery of cosmic expansion in 1929, Einstein referred to the introduction of Λ as his "greatest blunder"

The Cosmological Constant II

Introduction of $\Lambda \rightarrow$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon(t) - \frac{\kappa c^2}{R_0^2}\frac{1}{a(t)^2} + \frac{\Lambda}{3}$$

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$ (no change)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P) + \frac{\Lambda}{3}$$

The Cosmological Constant III

Λ corresponds to an energy of:

$$\varepsilon_\Lambda = \frac{c^2}{8\pi G}\Lambda$$

While Λ can in principle produce a static Universe (a'=0), this solution is as unstable as a pencile balanced on its sharpened point…

But: Because of its negative pressure, a Λ-like component with an energy slightly different than that assumed by Einstein has recently become part of the currently favoured cosmology, since this may explain the observed acceleration (a''>0) of the Universe