Cosmology 1FA209

2015

Exercises to be solved in class

1. *Olbers' paradox: Why is the sky dark at night?*
Let's assume that the universe is static and of infinite extent. The number density of stars is $n_\star$, the average stellar luminosity $L_\star$ and the average area of a stellar disc $A_\star$.

a) What is the total flux observed at some arbitrary point in the Universe?

b) If you furthermore assume that each star is covering an angle $A_\star/r^2$ on the sky and effectively hides the light from an infinite number of more distant objects, it may no longer be appropriate to assume the line of sight to go to infinity. Instead, one should integrate the flux received from stars only out to a distance $D_{\rm max}$, for which the whole sky will be evenly tiled by stellar discs. Derive this distance $D_{\rm max}$ and calculate the integrated flux from all the visible stars out to that distance. Does this resolve the paradox?

c) Describe at least four possible ways to solve Olbers' paradox.

2. *"Fun" with the fluid equation: Density evolution.*
Use the fluid equation $\dot{\rho}c^2 + 3\frac{\dot{a}}{a}(\rho c^2 + p) = 0$ and the equation of state $p = w\rho c^2$ to derive a proportionality relation between density and scale factor in the case of

a) a radiation dominated Universe

b) a matter dominated Universe

c) a Universe dominated by a cosmological constant.

3. *"Fun" with the Friedmann equation: The age of the Universe*
Starting from the Friedmann equation, derive an expression for the age of the Universe $t(z)$, in the case of a standard cosmological model including the parameters $\Omega_{\rm rad}$, $\Omega_{\rm M}$, Ω_Λ, $\Omega_{\rm tot}$.

a) What does the relation for $t(z)$ reduce to in the case of an empty (Milne) Universe?

b) What does the relation for $t(z)$ reduce to in the case of an Einstein-de Sitter Universe?

c) What is the current age of the Universe if $\Omega_{\rm rad} = 8.4\times10^{-5}$, $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$, $\Omega_{\rm tot} \approx 1.0$? What was it at redshift $z = 1$?

4. *Cosmological distances*
Consider a light source at a redshift of $z = 3$ in an Einstein-de Sitter Universe.

a) How far has the light from this object traveled to reach us?

b) How distant is this object today?

c) How distant was this object at the epoch when the light was emitted?

d) How far would light emitted from us today have to travel to reach the object?

5. *Standard candles*
Estimate the expected apparent magnitude $m_{\rm B}$ of a supernova type Ia (absolute magnitude $M_{\rm B} \approx -19.6$) at a redshift of $z = 0.8$ in a Universe described by $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$, $H_0 = 72$ km s^{-1} Mpc^{-1}. Cosmological k-corrections and dust effects may be neglected.

6. *Matter-radiation equality*
Estimate the redshift and temperature of the photon gas at the transition from radiation to matter domination, assuming a $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0$ cosmology.

7. *Dark energy and the big rip*
If the dark energy has an equation of state $w < -1$, the Universe may be ripped apart as the scale factor $a \to \infty$ when $t \to t_{\rm Rip}$. Derive an analytical expression for $t_{\rm Rip}$, under the assumption that the Universe has a flat geometry and is currently dominated by dark energy with constant w. Predict the time remaining before the Big Rip, in scenarios where:

a) $w = -1.1$
b) $w = -1.5$
c) $w = -2.0$

8. *Dark matter I*
Some unknown scientist launches a new theory about the nature of dark matter, in which the dark matter mainly consists of self-reproducing robots designed several billion years ago by some advanced civilization eager to explore the Universe. Ever since the demise of their masters in an unforeseen supernova explosion, the robots have been on a cosmic rampage, turning anything they can get their hands on into new robots. The founder of this theory estimates that essentially all the matter of the universe should now be in the form of such robot probes. Apparently, he has also located a broken specimen of this robot population buried in his own backyard. He is however not prepared to show this piece of alien technology to the public just yet. At a conference, he is confronted with the question: "But what are these robots made of?"

He thinks about this for a long time and then replies:

"Metal... Yeah, and some plastic."

Try to estimate an upper limit to the contribution of self-reproducing robots, $\Omega_{\rm robots}$ to the cosmological density of the Universe, assuming the standard Big Bang scenario to be correct.

9. *Dark matter II*
An estimated upper limit to the mass of the neutrino is 10 eV. Would this be enough to explain the estimated amount of dark matter in the Universe? What would the upper limit to the neutrino mass be if $\Omega_{\rm M} = 0.3$?

10. *Hunting for high-redshift objects*
The search for the very first generation of stars (the so-called population III) represents something of a holy grail in extragalactic astronomy. These objects are predicted to be very massive, short-lived and probably exclusively confined to the high-redshift Universe. Their exact masses are, however, still highly uncertain, and this introduces large uncertainties when assessing the prospects of detecting these stars. Estimate the number of 500, 50 and 5 $M_\odot$ population III stars at $z \approx 10$ that are expected to lie within 2.2 arcmin $\times$ 2.2 arcmin in the sky (approximately the field of view of both the existing *Hubble Space Telescope* and the upcoming *James Webb Space Telescope*), assuming a cosmic star formation rate for isolated population III stars of $\sim 10^{-5}$ $M_\odot$ cMpc^{-3} yr^{-1} at $z \approx 10$. A population III star of mass of 500 $M_\odot$ is predicted to have a lifetime of $t_{\rm life} \approx 2\times10^6$ yr, whereas 50 $M_\odot$ implies $t_{\rm life} \approx 4\times10^6$ yr and 5 $M_\odot$ as much as $t_{\rm life} \approx 6\times10^7$ yr.

11. *Big bang nucleosynthesis*
When the reactions that convert protons into neutrons fall out of equilibrium ($t_{\text{universe}} \sim 1$ s old), $\frac{n_n}{n_p} \approx 0.18$. Some of the neutrons will however decay back into protons before the formation of ^{4}He. Estimate the mass fraction of ^{4}He formed if this is assumed to take place when $t_{\text{universe}} \approx 180$ s.

12. *The flatness problem I.*
Use the Friedmann equation

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} - \frac{\kappa c^2}{R_0^2 a^2}$$

to derive

$$\Omega_{\text{tot}}(t) - 1 \propto \frac{1}{a(t)^2 H(t)^2}$$

13. *Size-redshift relation*
For flat cosmologies, the angular size Θ of an object with intrinsic linear size l (perpendicular to the line of sight) at redshift z is given by (small Θ)

$$\Theta = \frac{l}{D_{\text{A}}}$$

where D_{A} is the angular-size distance:

$$D_{\text{A}} = \frac{c}{H_0(1+z)} \int_1^{1+z} \frac{dx}{(\Omega_{\text{M}} x^3 + (1 - \Omega_{\text{M}} - \Omega_\Lambda)x^2 + \Omega_\Lambda)^{1/2}}.$$

a) Make a plot of $\Theta(z)$ for a $\Omega_{\text{M}} = 0.3$, $\Omega_\Lambda = 0.7$ universe (the integration will have to be done numerically, e.g. in Matlab). Comment on your (counterintuitive?) results.

b) What is the angular size of a galaxy at $z = 1$ with an intrinsic size of 10^5 ly? What would the angular size of the same object be if it was located at $z = 10$?

Hand-in exercises (deadline Jan 22, 2016)

1. *Hubble's law and luminosity distance*
A galaxy is observed at a redshift of $z = 0.25$. How distant is this object according to Hubble's law? How accurate is Hubble's law for estimating the luminosity distance at this redshift, under the assumption of a cosmological model with $\Omega_{\rm M} = 0.3$ and $\Omega_\Lambda = 0.7$?

2. *Fate of the Universe*
Starting from the Friedmann equation

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} - \frac{\kappa c^2}{R_0^2 a^2}, \quad (1)$$

demonstrate that a currently expanding, matter-only universe will continue to expand forever if $\Omega_{\rm M} \leq 1$, but not if $\Omega_{\rm M} > 1$.

3. *The era of dark-energy domination*
Estimate the redshift at which the Universe became dark-energy dominated, assuming $\Omega_{\rm M} = 0.3$ and $\Omega_{\rm DE} = 0.7$ today, and that the dark energy has an equation of state ($p = wc^2\rho$):
a) $w = -1.0$ (i.e. a cosmological constant)
b) $w = -1.5$

4. *Dark energy and supernovae type Ia*
The redshifts and apparent magnitudes of a small sample of supernovae type Ia are listed in Table 1. Use this data to determine which of the following three cosmological models is the most likely:
a) $\Omega_{\rm M} = 1.0$, $\Omega_\Lambda = 0.0$
b) $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$
c) $\Omega_{\rm M} = 0.5$, $\Omega_\Lambda = 0.5$
You can assume $H_0 = 72$ km s^{-1} Mpc^{-1} and an absolute magnitude for type Ia supernovae of $M_B = -19.3$ in all cases. Both dust extinction corrections and cosmological k-corrections may be ignored. The estimated photometric 1σ errors of the measurements are $\sigma_m = 0.1$ magnitudes. Errors in the redshift determinations can be assumed negligible. Note: As this is just a toy example with artificial data, you should not expect the concordance model to win by default.

5. *Cosmic Microwave Background Radiation II*
The CMBR can be very well represented by a black-body spectrum with a current temperature of 2.728 K. Use this information to estimate its contribution $\Omega_{\rm CMBR}$ to the cosmological density.

6. *The flatness problem II*
If $|\Omega_{\rm tot} - 1| < 0.1$ now, what value of $|\Omega_{\rm tot} - 1|$ does that imply at
a) the epoch of matter-radiation equality?
b) the Planck time, assuming no inflation?
In both-cases, late-time domination by dark energy may be neglected.

Table 1: Supernova Type Ia data

z	m_B
0.022	15.51
0.041	16.93
0.057	17.68
0.083	18.64
0.11	19.11
0.35	21.94
0.42	22.41
0.57	23.19
0.83	23.94
0.90	24.27
0.98	24.49
1.10	24.62
1.30	25.12
1.50	25.67
1.90	26.15

Preliminary schedule for exercise classes
Remember that it will be much easier to understand the solutions presented during the exercise sessions if you have already attempted to solve the problems yourself.

Exercise session 1 (Nov 24) - Problems 1–6
Exercise session 2 (Dec 4) - Problems 7–13

Erik Zackrisson, October 2015