Cosmology 2016

Exercises with solutions – batch II

1. *Dark energy and the big rip*
If the dark energy has an equation of state $w < -1$, the Universe may be ripped apart as the scale factor $a \to \infty$ when $t \to t_{\text{Rip}}$. Derive an analytical expression for t_{Rip}, under the assumption that the Universe has a flat geometry and is currently dominated by dark energy with constant w. Predict the time remaining before the Big Rip, in scenarios where:
a) $w = -1.1$
b) $w = -1.5$
c) $w = -2.0$

Solution:
In exercise 1 of exercise batch I, we concluded that the energy density of cosmological energy components evolve with scale factor a in the following way:

$$\epsilon \propto a^{-3(1+w)}. \tag{1}$$

Please note that for dark energy components with $w < -1$, this implies that the energy density of dark energy actually increases as the Universe expands ("phantom energy"!). In exercise 2 of batch I, we also derived the following form of the Friedmann equation:

$$\frac{H(t)^2}{H_0^2} = \frac{\epsilon(t)}{\epsilon_{\text{c},0}} + \frac{1 - \Omega_{\text{tot},0}}{a(t)^2}. \tag{2}$$

Assuming $\Omega_{\text{tot},0} = 1$ (flat geometry) and that the energy density of the Universe is dominated by dark energy ($\epsilon \approx \epsilon_{\text{DE}}$, we can insert (1) into (2) to get:

$$\frac{H(t)^2}{H_0^2} = \frac{\epsilon_{\text{DE},0} a^{-3(1+w)}}{\epsilon_{\text{c},0}} = \Omega_{\text{DE},0} a^{-3(1+w)}. \tag{3}$$

Now, let's use the same trick as in exercise 2 of batch I to extract an explicit time dependence... First recall that the Hubble parameter is defined as:

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt}. \tag{4}$$

Inserting (4) into (3) gives:

$$\frac{da}{dt} = H_0 \left(\Omega_{\text{DE},0}\right)^{1/2} a^{-\frac{3w}{2} - \frac{1}{2}}. \tag{5}$$

Rearrangement of (5) now leaves us with:

$$dt = \frac{da}{H_0 \left(\Omega_{\text{DE},0}\right)^{1/2} a^{-\frac{3w}{2} - \frac{1}{2}}}. \tag{6}$$

The time of the Big Rip, t_{Rip} is defined as the time when $a \to \infty$, so let's integrate both sides of (6) and use $t = t_{\text{Rip}}$ and $a = \infty$ as upper integration limits:

$$\int_{t_0}^{t_{\text{rip}}} dt = \frac{1}{H_0 \left(\Omega_{\text{DE},0}\right)^{1/2}} \int_{a_0=1}^{\infty} \frac{da}{a^{-\frac{3w}{2} - \frac{1}{2}}}. \tag{7}$$

If we solve this, we get:

$$t_{\text{rip}} - t_0 = \frac{2}{3H_0(\Omega_{\text{DE},0})^{1/2}(w+1)} \left[a^{\frac{3}{2}(w+1)}\right]_1^{\infty} \tag{8}$$

In the case of $w < -1$, this simlifies to:

$$t_{\rm rip} - t_0 = -\frac{2}{3H_0(\Omega_{\rm DE,0})^{1/2}(w+1)} \tag{9}$$

Plugging in the numbers: If we adopt $H_0 = 70$ km s^{-1} Mpc^{-1} and $\Omega_{\rm DE,0} = 1.0$ (the latter may seem a bit stupid, but otherwise the Universe would not be flat given our assumptions), we can calculate $t_{\rm rip} - t_0$, i.e. the time remaining until the Big Rip as:
a) $w = -1.1 \rightarrow t_{\rm rip} - t_0 \approx 93$ Gyr
b) $w = -1.5 \rightarrow t_{\rm rip} - t_0 \approx 19$ Gyr
c) $w = -2.0 \rightarrow t_{\rm rip} - t_0 \approx 9$ Gyr

2. *Big bang nucleosynthesis*
When the reactions that convert protons into neutrons fall out of equilibrium ($t_{\rm universe} \sim 1$ s old), $\frac{n_n}{n_p} \approx 1/5$. Some of the neutrons will however decay back into protons before the formation of ^{4}He. Estimate the mass fraction of ^{4}He formed if this is assumed to take place when $t_{\rm universe} \approx 180$ s.

Solution: In astrophysics, the baryonic mass fraction locked up in helium is usually denoted Y. If we make the approximation that the masses of the nucleons (protons and neutrons) are equal, then the primordial abundance of ^{4}He, can be expressed as:

$$Y(^4{\rm He}) \approx \frac{4 n_{^4{\rm He}}}{n_{\rm n} + n_{\rm p}}, \tag{10}$$

where $n_{^4{\rm He}}$ is the number density of ^{4}He nuclei, $n_{\rm n}$ is the number density of neutrons and $n_{\rm p}$ is the number density of protons. If we make the simplifying assumption that all neutrons end up in ^{4}He, (10) can be rewritten as:

$$Y(^4{\rm He}) \approx \frac{4(n_{\rm n}/2)}{n_{\rm n} + n_{\rm p}} = \frac{2(n_{\rm n}/n_{\rm p})}{1 + (n_{\rm n}/n_{\rm p})}. \tag{11}$$

The $(n_{\rm n}/n_{\rm p})$ ratio is fixed at $(n_{\rm n}/n_{\rm p}) \approx 0.2$ at the point where neutrinos decouple from neutrons and protons (~ 1 s after the Big Bang). If all the ^{4}He nuclei would from at this time, we would end up with a primordial ^{4}He fraction of $Y(^4{\rm He}) \approx 2 \times 0.2/(1 + 0.2)$ or ≈ 0.33. However, most of the ^{4}He nuclei do not form quite this early, and some of the neutrons will therefore decay back into protons before ^{4}He have time to form.

If we use $n_{\rm n,0}$ to denote the neutron density directly after the point where neutrinon freeze-out of reactions with neutrons and protons, the number density $n_{\rm n}$ at some later time t can be expressed as:

$$n_{\rm n} = n_{\rm n,0} \exp(-t/\tau), \tag{12}$$

where τ is the mean lifetime of free neutrons. When the neutrons decay, they decay into protons, so the number density of protons $n_{\rm p}$ at time t is boosted compared to the number density $n_{\rm p,0}$ at freeze-out:

$$n_{\rm p} = n_{\rm p,0} + n_{\rm n,0}(1 - \exp(-t/\tau)). \tag{13}$$

To derive the neutron-to-proton ratio at time t we simply divide (12) by (13). After some rearranging, we get:

$$\frac{n_{\rm n}}{n_{\rm p}} = \frac{1}{\exp(t/\tau)(1 + n_{\rm p,0}/n_{\rm n,0}) - 1}. \tag{14}$$

Plugging in the numbers: The mean lifetime of free neutrons is about 15 minutes, and modern estimates place it at $\tau \approx 888$ s (there is some controversy regarding the exact value, but it would not have any significant impact on our calculations). If we adopt $(n_{\rm p,0}/n_{\rm n,0}) \approx 1/0.2 = 5$ and assume that all ^{4}He instantaneously forms 180 s after the Big Bang (at $t \approx 180 - 1 = 179$ s after neutrinos stop talking to protons and neutrons), (14) gives $Y(^4{\rm He}) \approx 0.26$, which is somewhat lower than the $Y(^4{\rm He}) \approx 0.33$ ratio we would get if neutron decay were not considered.

Comment: In real life, ^{4}He isn't instantaneously formed, and the freeze-out ratio isn't exactly $n_{\rm n,0}/n_{\rm p,0} = 0.2$. More sophisticated calculations therefore tend to favour a primordial ^{4}He mass fraction of $Y(^4{\rm He}) \approx 0.25$, which is consistent with modern measurements of the Helium abundance in near-pristine gas.

3. *The size-redshift relation*
Assuming an Einstein-de Sitter Universe, how many arcseconds would a galaxy with diameter 3 kpc span in the sky, assuming that it is located at redshift **a)** $z = 1$?; **b)** $z = 10$?

Solution:
The relation between the linear size l of an object at redshift z and the angle Θ it subtends in the sky is given by (7.36 in the textbook):

$$\Theta = \frac{l}{d_{\rm A}(z)}, \tag{15}$$

where $d_{\rm A}(z)$ is the angular diameter distance to the object. The angular diameter distance is related to the luminosity distance $d_{\rm L}(z)$ through the following relation (7.37 in the textbook):

$$d_{\rm A}(z) = \frac{d_{\rm L}(z)}{(1+z)^2}. \tag{16}$$

Inserting (16) into (15) gives:

$$\Theta = \frac{l(1+z)^2}{d_{\rm L}(z)}. \tag{17}$$

Now, recall the expression for the luminosity distances for a flat Universe from exercise 4 in batch I:

$$d_{\rm L} = (1+z)\frac{c}{H_0}\int_0^z \frac{dz}{\left[\Omega_{\rm M}(1+z)^3 + \Omega_\Lambda\right]^{1/2}}. \tag{18}$$

In the Einstein-de Sitter Universe where $\Omega_\Lambda = 0$ and $\Omega_{\rm M} = 1$, this simplifies to:

$$d_{\rm L} = (1+z)\frac{c}{H_0}\int_0^z \frac{dz}{(1+z)^{3/2}}. \tag{19}$$

If you solve this integral, you get:

$$d_{\rm L} = (1+z)\frac{2c}{H_0}\left[1-(1+z)^{-1/2}\right]. \tag{20}$$

Inserting (20) into (17) gives:

$$\Theta = \frac{l\ H_0(1+z)}{2c\left[1-(1+z)^{-1/2}\right]}. \tag{21}$$

Now, Θ will in this case be given in radians. If you want it in arcseconds, you need to use the following conversion:

$$\Theta_{\rm arcsec} = 1.296\times10^6\frac{\Theta}{2\pi}, \tag{22}$$

where the numerical factor 1.296×10^6 stems from the fact that a full circle spans 360 degrees and each degree contains 3600 arcseconds ($360 \times 3600 = 1.296 \times 10^6$). If we apply (22) to (21) we arrive at the final expression:

$$\Theta_{\rm arcsec} = \frac{1.296\times10^6\ l\ H_0(1+z)}{4\pi c\left[1-(1+z)^{-1/2}\right]}. \tag{23}$$

Plugging in the numbers:
Using $l = 3$ kpc, H_0=70 km s^{-1} Mpc^{-1}, $c = 3 \times 10^5$ km s^{-1} in (23) gives:
a) $\Theta_{\rm arcsec} \approx 0.49$ arcsec for $z = 1$
b) $\Theta_{\rm arcsec} \approx 1.1$ arcsec for $z = 10$

Comment: This illustrates that, quite counter-intuitively, an expanding Universe, an object with fixed linear size can appear larger in the sky if it is located further away! For this particular cosmology, objects with a fixed linear size will appear to shrink in angular size out to a redshift of $z \approx 1.25$, and then start to grow at even greater redshifts.

4. *The flatness problem I.*
Use the Friedmann equation

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3c^2}\epsilon_{\text{tot}}(t) - \frac{\kappa c^2}{R_0^2 a(t)^2}$$

to derive

$$\Omega_{\text{tot}}(t) - 1 \propto \frac{1}{a(t)^2 H(t)^2}$$

Solution:
At any epoch, $\Omega_{\text{tot}}(t)$ can be described as (4.28 in the textbook):

$$\Omega_{\text{tot}}(t) = \frac{\epsilon_{\text{tot}}(t)}{\epsilon_{\text{c}}(t)}, \tag{24}$$

with $\epsilon_{\text{c}}(t)$ given by (4.25 in the textbook):

$$\epsilon_{\text{c}}(t) = \frac{3c^2}{8\pi G} H(t)^2. \tag{25}$$

If we combine (24) and (25), we get:

$$\epsilon_{\text{tot}}(t) = \frac{3c^2}{8\pi G} H(t)^2 \Omega_{\text{tot}}(t). \tag{26}$$

If we insert (26) into the Friedmann equation

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G}{3c^2}\epsilon_{\text{tot}}(t) - \frac{\kappa c^2}{R_0^2 a(t)^2}, \tag{27}$$

we arrive at:

$$\frac{\dot{a}}{a})^2 = H(t)^2 \Omega_{\text{tot}}(t) - \frac{\kappa c^2}{R_0^2 a(t)^2}. \tag{28}$$

Now, recall that

$$H(t) = \frac{\dot{a}}{a}, \tag{29}$$

and use this to rewrite (28). We then get, after rearranging the terms:

$$\Omega_{\text{tot}}(t) - 1 = \frac{\kappa c^2}{R_0^2 a(t)^2 H(t)^2}. \tag{30}$$

Since $\frac{\kappa c^2}{R_0^2}$ is just a constant, (30) implies that

$$\Omega_{\text{tot}}(t) - 1 \propto \frac{1}{a(t)^2 H(t)^2}. \tag{31}$$

And there you have it!
Hint: It is entirely possible that (30) could turn out to be quite handy when solving hand-in exercise 6 (The flatness problem II)...

Erik Zackrisson, November 2016