Cosmology 2016

Exercises with solutions – batch I

1. *Learning to use the fluid equation: The density evolution of the Universe*
Use the fluid equation $\dot{\epsilon}+3\frac{\dot{a}}{a}(\epsilon+P)=0$ and the equation of state $P=w\epsilon$ to derive a proportionality relation between mass density and scale factor in the case of

a) a radiation dominated Universe
b) a matter dominated Universe
c) a Universe dominated by a cosmological constant.

Solution: The ϵ in the fluid equation

$$\dot{\epsilon}+3\frac{\dot{a}}{a}(\epsilon+P)=0 \tag{1}$$

and the equation of state

$$P=w\epsilon \tag{2}$$

represents *energy* density. We are, however, here asked to derive the relation between *mass* density ρ and the scale factor a. Luckily, the conversion between energy density and mass density is simply a question of applying $\epsilon=\rho c^2$ (think $E=mc^2$ and divide both sides by the volume), so you can safely carry out all the required algebra with the energy density versions of (1) and (2) and apply the conversion between ϵ and ρ at the very end.

Let's get cracking! By inserting (2) into (1), you get:

$$\dot{\epsilon}=-3(1+w)\frac{\dot{a}}{a}\epsilon \tag{3}$$

Hmm... You wanted ϵ as a function of a, but this expression also contains $\dot{\epsilon}$ and $\dot{a}$, so how does this help exactly? There is a standard trick you can apply in situations like these! First rewrite (3) by moving ϵ and $\dot{\epsilon}$ to the left-hand side:

$$\frac{\dot{\epsilon}}{\epsilon}=-3(1+w)\frac{\dot{a}}{a}. \tag{4}$$

Now integrate both sides (recalling that $\ln f(x)$ is the primitive function of $f'(x)/f(x)$):

$$\ln\epsilon=-3(1+w)\ln a+C_1, \tag{5}$$

where C_1 is a constant. Now exponentiate both sides:

$$\epsilon=\exp(-3(1+w)\ln a)\cdot\exp(C_1) \tag{6}$$

This may seem even more complicated than before, but there are some simplifications to be made:

$$\exp(-3(1+w)\ln a)=\exp(\ln a^{-3(1+w)})=a^{-3(1+w)}, \tag{7}$$

and $\exp(C_1)$ is just a constant anyway (let's call it C_2), so (6) simplifies to:

$$\epsilon=C_2a^{-3(1+w)}. \tag{8}$$

Because of the simple $\epsilon=\rho c^2$ conversion, the proportionality relation between mass density ρ and scale factor then becomes:

$$\rho\propto a^{-3(1+w)}. \tag{9}$$

Now, we're finally in good shape to address the detailed density evolution of specific components of the Universe.

a) According to statistical mechanics, radiation has an equation of state parameter $w = 1/3$, so the proportionality relation (9) between mass density and scale factor for radiation-like components of the Universe becomes:

$$\rho_{\rm rad} \propto a^{-4}. \tag{10}$$

b) Non-relativistic matter exhibits zero pressure, so (2) implies $w = 0$. Expression (9) then reduces to:

$$\rho_{\rm M} \propto a^{-3} \tag{11}$$

c) A cosmological constant Λ has $w = -1$, which means that (9) reduces to:

$$\rho_{\Lambda} \propto a^{0} \tag{12}$$

In other words, ρ_Λ does not change with a (or time) - the mass/energy density of this component stays the same even though the Universe is expanding!
Comment: The scale factor a gets bigger as the Universe grows. Hence, the density of both radiation (10) and matter (11) get diluted with the expansion of the Universe, whereas the cosmological constant density (12) remains the same (that's why it's called a "constant"). Hence, even if matter and radiation initially rules the Universe, Λ will become the dominant component over time.

2. *Learning to use the Friedmann equation: The age of the Universe*
Starting from the Friedmann equation, derive an expression for the age of the Universe $t(z)$, in the case of a standard cosmological model including the parameters $\Omega_{\rm rad}$, $\Omega_{\rm M}$, Ω_Λ, $\Omega_{\rm tot}$.

a) What does the relation for $t(z)$ reduce to in the case of an Einstein-de Sitter Universe?

b) What is the current age of the Universe if $\Omega_{\rm rad} = 8.4\times10^{-5}$, $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$, $\Omega_{\rm tot} \approx 1.0$?

Solution: When faced with this problem, many students immediately give up, since the Friedmann equation (4.19 in the textbook) does not appear to contain any explicit time variable t

$$H(t)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{\kappa c^2}{R_0^2 a(t)^2}. \tag{13}$$

Luckily, there are time dependences in the Friedmann equation that you can exploit to extract t – but it will take quite a bit of algebra to get there.

One can show that the following relation (see 4.31 in the textbook) holds between $\Omega_{\rm tot}$ (where $\Omega_{\rm tot}$ is the sum of the Ωs that you care to consider in your cosmological model: $\Omega_{\rm tot} = \Omega_\Lambda + \Omega_{\rm M} + \Omega_{\rm rad}...$) and the cosmic curvature κ:

$$\frac{\kappa}{R_0^2} = \frac{H_0^2}{c^2}(\Omega_{\rm tot} - 1) \tag{14}$$

Inserting (14) into (13) gives:

$$H(t)^2 = \frac{8\pi G}{3c^2}\epsilon(t) - \frac{H_0^2}{a(t)^2}(\Omega_{\rm tot,0} - 1). \tag{15}$$

Now, recall that the critical energy density (4.25 in the textbook) is given by

$$\epsilon_{\rm c} = \frac{3c^2}{8\pi G}H(t)^2 \tag{16}$$

Divide (15) by H_0^2 on both sides and use (16) $\Rightarrow$

$$\frac{H(t)^2}{H_0^2} = \frac{\epsilon(t)}{\epsilon_{\rm c,0}} + \frac{1 - \Omega_{\rm tot,0}}{a(t)^2} \tag{17}$$

At this point, you can split $\epsilon(t)$ into as many energy components that you care to consider in your cosmological model. In this case, we'll settle for radiation, matter and a cosmological constant Λ:

$$\epsilon(t) = \epsilon_{\rm rad}(t) + \epsilon_{\rm M}(t) + \epsilon_\Lambda(t) \tag{18}$$

Inserting (18) into (17) then gives:

$$\frac{H(t)^2}{H_0^2} = \frac{\epsilon_{\rm rad}(t) + \epsilon_{\rm M}(t) + \epsilon_\Lambda(t)}{\epsilon_{\rm c,0}} + \frac{1 - \Omega_{\rm tot,0}}{a(t)^2} \tag{19}$$

As derived in the solution to exercise 1, the three components we consider exhibit the following scale factor dependencies:

$$\epsilon_{\rm rad} \propto a^{-4}. \tag{20}$$

$$\epsilon_{\rm M} \propto a^{-3} \tag{21}$$

$$\epsilon_\Lambda \propto a^0 \tag{22}$$

If we insert (20), (21) and (22) into (19) and exploit $\Omega_i = \epsilon_i/\epsilon_{\rm c}$, we get:

$$\frac{H(t)^2}{H_0^2} = \Omega_{\rm rad,0}a^{-4} + \Omega_{\rm M,0}a^{-3} + \Omega_{\Lambda,0} + (1 - \Omega_{\rm tot,0})a^{-2} \tag{23}$$

This is all very nice, but there is still no explicit time dependence! But here comes the magic:

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{a}\frac{da}{dt} \tag{24}$$

If you use this relation for $H(t)$ and insert it into (23), we get

$$\frac{da}{dt} = aH_0 \left(\Omega_{\rm rad,0}a^{-4} + \Omega_{\rm M,0}a^{-3} + \Omega_{\Lambda,0} + (1 - \Omega_{\rm tot,0})a^{-2}\right)^{1/2} \tag{25}$$

In general:

$$t(a) = \int_0^a \frac{dt}{da}da \tag{26}$$

If we take the inverse of da/dt in (25) we get dt/da which we can insert into (26), with the result:

$$t(a) = \frac{1}{H_0}\int_0^a \frac{a}{\left(\Omega_{\rm rad,0}a^{-2} + \Omega_{\rm M,0}a^{-1} + \Omega_{\Lambda,0}a^2 + (1 - \Omega_{\rm tot,0})\right)^{1/2}} \tag{27}$$

This is $t(a)$, but we need $t(z)$. To make the transition, we need to realize that

$$da = \frac{da}{dz}dz \tag{28}$$

and

$$a = \frac{1}{1+z} \Rightarrow \frac{da}{dz} = -\frac{1}{(1+z)^2} \tag{29}$$

If we insert this into (27) and transform the integration limits, we arrive at the final expression:

$$t(z) = \frac{1}{H_0}\int_z^\infty \frac{dz}{(1+z)\left[\Omega_{\rm rad,0}(1+z)^4 + \Omega_{\rm M,0}(1+z)^3 + \Omega_{\Lambda,0} + (1 - \Omega_{\rm tot,0})(1+z)^2\right]^{1/2}}. \tag{30}$$

a) In the Einstein-de Sitter Universe, we have $\Omega_{\rm rad,0} = 0$, $\Omega_{\rm M} = 1.0$, $\Omega_\Lambda = 0 \Rightarrow \Omega_{\rm tot} = 1$. In this case, (30) reduces to:

$$t(z) = \frac{1}{H_0}\int_z^\infty \frac{dz}{(1+z)^{5/2}} = \frac{2}{3H_0^{3/2}} \tag{31}$$

At the current epoch ($z = 0$), the age of the Universe in the Einstein-de Sitter Universe then becomes:

$$t_0 = \frac{2}{3H_0} \tag{32}$$

All you need to do is to plug your adopted value of H_0 into (32), but unless you first convert H_0 from [km s^{-1} Mpc^{-1}] into [s^{-1}], the resulting units will be pretty hard to interpret.

$H_0 = 70$ km s^{-1} Mpc^{-1} $= 70 \cdot 1000/(10^6 \cdot 3.086 \times 10^{16}) = 2.2683 \times 10^{-18}$ s^{-1}

When H_0 in these units is applied to (32) we get $t_0 \approx 2.939 \times 10^{17}$ s or ≈ 9.3 Gyr

b) To solve (30) for the Benchmark model, you need to use numerical integration. On the course hompage, there is a small Matlab script that allows you do this. For $\Omega_{\rm rad} = 8.4 \times 10^{-5}$, $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$, $\Omega_{\rm tot} \approx 1.0$, we get $t_0 \approx 13.1$ Gyr. This is somewhat smaller than the 13.8 Gyr often quoted in the literature, but this is because the currently favoured cosmological parameters are slightly different from the ones used here.

3. *Cosmological distances*
Consider a light source at a redshift of $z = 3$ in an Einstein-de Sitter Universe.
a) How far has the light from this object traveled to reach us?
b) How distant is this object today?

Solution:
a) The distance experienced by a photon on its path towards us is given by the so-called distance by light-travel time D_c:

$$D_\mathrm{c} = c\Delta t, \tag{33}$$

where Δt is the time interval between the epoch when the light was emitted and the epoch when the light was detected. In our case:

$$\Delta t = t_0 - t(z), \tag{34}$$

where t_0 is the current age of the Universe and $t(z)$ is the age of the Universe when the light was emitted (at redshift z). Inserting (34) into (33) gives:

$$D_\mathrm{c} = c(t_0 - t(z)). \tag{35}$$

In an Einsten-de Sitter Universe (i.e. $\Omega_\mathrm{M} = 1.0$, $\Omega_\Lambda = 0.0$), the age of the Universe at redshift z is (see exercise 2b for derivation):

$$t(z) = \frac{2}{3H_0(1+z)^{3/2}}. \tag{36}$$

The age of the Universe at the current time t_0 then becomes:

$$t_0 = \frac{2}{3H_0}, \tag{37}$$

since the redshift is $z = 0$ at the current epoch. Inserting (36) and (37) into (35) gives us the final expression for the distance by light-travel time in an Einstein-de Sitter Universe:

$$D_\mathrm{c} = \frac{2c}{3H_0}(1 - (1+z)^{-3/2}). \tag{38}$$

Plugging in the numbers:
$H_0 = 70$ km s Mpc^{-1}, $z = 3$ in (38) $\Rightarrow D_\mathrm{c} \approx 2.5$ Gpc or ≈ 8.1 Gly.

b) The distance to this object today is given by the proper distance

$$d_\mathrm{p} = c\int_{t(z)}^{t_0} \frac{dt}{a(t)} \tag{39}$$

To calculate this, we need an expression for the time dependence of the scale factor $a(t)$. Luckily, (36), which describes $t(z)$ – the age of the Universe at redshift z – can quite easily be manipulated to give us $a(t)$. Recall that the definition fo the scale factor a is:

$$a = \frac{1}{1+z} \tag{40}$$

By inserting (40) into (36), we get:

$$t(z) = \frac{2a^{3/2}}{3H_0}. \tag{41}$$

Extracting a from (41) yields:

$$a(t) = \left(\frac{3H_0 t(z)}{2}\right)^{(2/3)} \tag{42}$$

Please note that in the currently favoured cosmology ($\Omega_{\mathrm{M}} \approx 0.3$, $\Omega_\Lambda \approx 0.7$), expression (42) would take on a *far more complicated* form.

Inserting (42) into (39) gives us:

$$d_{\mathrm{p}}(t_0) = c \left(\frac{2}{3H_0} \right)^{2/3} \int_{t(z)}^{t_0} t^{-2/3} dt \tag{43}$$

If you solve this integral, you get:

$$d_{\mathrm{p}}(t_0) = 3c \left(\frac{2}{3H_0} \right)^{(} 2/3)(t_0^{1/3} - t(z)^{1/3}]). \tag{44}$$

Finally, by inserting (37) and (36) into (44) we get the expression:

$$d_{\mathrm{p}}(t_0) = \frac{2c}{H_0} \left[1 - \frac{1}{(1+z)^{1/2}} \right] \tag{45}$$

Plugging in the numbers:
$H_0 = 70$ km s^{-1} Mpc^{-1} and $z = 3$ in (45) $\Rightarrow D_{\mathrm{c}} \approx 4.2$ Gpc or ≈ 14 Gly. Please note that this distance is considerably larger than the distance that light has actually travelled from this object to reach us. This happens because of the expansion of the Universe – the object has continued to receed further from us after the light we now receive was emitted.

4. *Standard candles*
Estimate the expected apparent magnitude $m_{\rm B}$ of a supernova type Ia (absolute magnitude $M_{\rm B} \approx -19.6$) at a redshift of $z = 0.8$ in a Universe described by $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$, $H_0 = 70$ km s^{-1} Mpc^{-1}. Cosmological k-corrections, band-pass corrections and dust effects may be neglected.

Solution: Eq. (7.50) in the textbook describes the relation between the apparent magnitude m and absolute magnitude M of a high-redshift light source:

$$m - M = 5\log_{10}\left(\frac{d_{\rm L}}{1\ {\rm Mpc}}\right) + 25, \tag{46}$$

where d_L is the luminosity distance. The luminosity distance for a Universe filled with non-relativistic matter and a cosmological constant Λ is given by:

$$d_{\rm L} = (1+z)\frac{c}{H_0}\int_0^z \frac{dz}{[\Omega_{\rm M}(1+z)^3 - (\Omega_{\rm M} + \Omega_\Lambda - 1)(1+z)^2 + \Omega_\Lambda]^{1/2}}. \tag{47}$$

This integral has no known analytical solution and needs to be solved numerically. On the course homepage, there's a link to a small piece of code that demonstrates how numerical integration can be done in Matlab (freely available for all Uppsala University students).

Plugging in the numbers:
Inserting $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$, $c = 3\times10^8$ m s^{-1}, $H_0 = 70$ km s^{-1} Mpc^{-1} into (47) gives $d_{\rm L} \approx 5$ Gpc. Inserting this luminosity distance into (46) and $M_{\rm B} = -19.6$ then gives $m_B = 23.8$.

Comment: The relation (46) is formally valid for bolometric magnitudes – i.e. magnitudes derived by integrating over the whole electromagnetic spectrum of the source. When we are consider the flux within a specific wavelength range (the B band in our case), additional corrections should be applied. The k-correction is usually defined to take both in-band effects (i.e. the fact taht you are not capturing the whole spectrum in your observations) and the redshifting of the spectrum (the fact that the spectral features you detect in your B filter were emitted at wavelengths that were smaller by a factor (1+z) when emitted). However, since the textbook does not treat such corrections at all, we will take (46) to suffice throughout this course - just remember to look into the k-correction when attempting something like this professionally.

5. *Matter-radiation equality*
Estimate the redshift at the transition from radiation to matter domination, assuming a $\Omega_\mathrm{M} = 0.3$, $\Omega_\Lambda = 0$ cosmology.

Solution: As derived in the solution to exercise 1, the radiation density and matter density exhibit the following scale factor dependencies:

$$\rho_\mathrm{rad} \propto a^{-4}. \tag{48}$$

$$\rho_\mathrm{M} \propto a^{-3} \tag{49}$$

These can be converted from $\rho(a)$ to $\rho(z)$ using

$$a = \frac{1}{1+z}. \tag{50}$$

By inserting (50) into (48) and (49), and scaling from the present-day densities $\rho_\mathrm{rad,0}$ and $\rho_\mathrm{M,0}$ we get:

$$\rho_\mathrm{rad} = \rho_\mathrm{rad,0}(1+z)^4. \tag{51}$$

$$\rho_\mathrm{M} = \rho_\mathrm{M,0}(1+z)^3 \tag{52}$$

In this exercise, we are interested in the transition from radiation to matter domination, i.e. the epoch when $\rho_\mathrm{M} = \rho_\mathrm{rad}$, so let's explore when this equality is met by setting (51) equal to (52):

$$\rho_\mathrm{rad} = \rho_\mathrm{M} \Rightarrow \rho_\mathrm{rad,0}(1+z)^4 = \rho_\mathrm{M,0}(1+z)^3. \tag{53}$$

If you extract z, you then get:

$$z_\mathrm{eq} = \frac{\rho_\mathrm{M,0}}{\rho_\mathrm{rad,0}} - 1. \tag{54}$$

Now, remember that $\Omega = \rho/\rho_\mathrm{c}$ and use this to rewrite (54) $\Rightarrow$

$$z_\mathrm{eq} = \frac{\Omega_\mathrm{M,0}}{\Omega_\mathrm{rad,0}} - 1. \tag{55}$$

Plugging in the numbers:
$\Omega_\mathrm{M} = 0.3$, $\Omega_\mathrm{rad} \approx 8.4 \times 10^{-5}$ in (55) $\Rightarrow z_\mathrm{eq} \approx 3570$.
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