Physics of Galaxies 2018

Exercises with solutions – Batch II

1. *Surface brightness*
Derive an approximate expression for the total luminosity of a disk galaxy with a exponentially declining surface brightness distribution given by $I(r) = I_0 \exp(-\frac{r}{a})$, where a represents the scale length.

Solution: Surface brightness is usually expressed as I (units $L_\odot$ pc^{-2}) or μ (units mag arcsec^{-2}). As counter-intuitive as this may seem, surface brightness is actually independent of distance at low redshift (although demonstrating this requires a somewhat cumbersome calculation). Beyond the local Universe (at $z > 0.05$ or so), however, an effect known as redshift dimming sets in and dilutes surface brightness by a factor $(1 + z)^{-4}$.

For the problem at hand, consider the face-on disk depicted in Figure 1, for which the surface brigtness at radius r is given by:

$$I(r) = I_0 \exp(-\frac{r}{a}), \tag{1}$$

where I_0 is the central surface brightness (i.e. at $r = 0$) and a is the disk scale length.

In the case of a disk with constant surface brightness ($I(r) = I_0$), you would simply multiply I_0 (in $L_\odot$ pc^{-2}) by the total area of the disk $\pi r_{\rm max}^2$ (in pc^2) to get the luminosity L (in $L_\odot$). However, since the surface brightness changes across the disk, you have integrate over r:

$$L = \int_0^{r_{\rm max}} 2\pi r I(r)\ \mathrm{d}r. \tag{2}$$

Here, $2\pi r$ is simply the circumference of a circle with radius r.

To understand why eq.(2) makes sense, you can think of the disk as divided into a number of annuli, each with thickness dr (as illustrated in Figure 1). The area of each such annulus is approximately $2\pi r$ multiplied by dr, and the luminosity of each annulus would then be $I(r)2\pi r\mathrm{d}r$. A summation over all annuli would give you a rough estimate of the luminosity:

$$L \approx \sum_i 2\pi r_i I_i \mathrm{d}r_i. \tag{3}$$

However, doing the proper integration in eq.(2) is more accurate.

Inserting eq.(1) into eq.(2) gives:

$$L = \int_0^{r_{\rm max}} 2\pi I_0 r \exp(-\frac{r}{a})\ \mathrm{d}r. \tag{4}$$

Even in the case of an infinite exponential disk ($r_{\rm max} = \infty$), the resulting luminosity is finite. Eq.(4) becomes:

$$L = 2\pi I_0 \int_0^{\infty} r \exp(-\frac{r}{a})\ \mathrm{d}r. \tag{5}$$

This is a standard integral (general form $\int x \exp(cx)\ \mathrm{d}x$), which you can look up in integral tables. The result is:

$$L = 2\pi I_0 \int_0^{\infty} r \exp(-\frac{r}{a})\ \mathrm{d}r = 2\pi I_0 \left[a^2 \exp(-\frac{r}{a})(-\frac{r}{a} - 1)\right]_0^{\infty} = 2\pi I_0(0 + a^2) = 2\pi I_0 a^2. \tag{6}$$

Even though real disk galaxies are not infinite, eq.(6) often provides a good estimate of the luminosity, since the faint outskirts of the disk actually contribute very little to the overall integral.

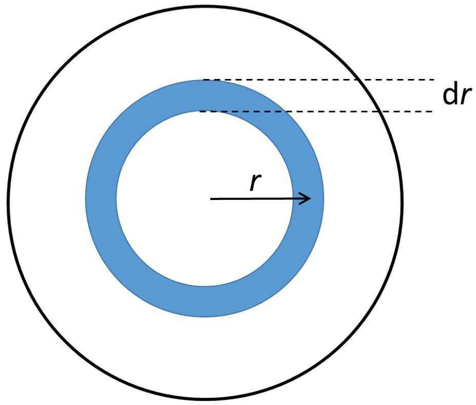

Figure 1: Schematic illustration of a face-on disk.

2. *Dust corrections*
A star-forming galaxy is observed to have an apparent R-band magnitude of $m_R = 23.0$ mag, an Hα flux of $f_{\mathrm{H}\alpha} = 1.0 \times 10^{-15}$ erg s^{-1} cm^{-2} and a flux ratio between the Hα and Hβ emission lines of Hα/H$\beta = 3.5$. Use this information to correct the R-band magnitude and the Hα flux for dust attenuation within the galaxy.

Solution: The so-called Balmer decrement – the observed ratio of the fluxes of the Hα and Hβ emission lines – serves as a very convenient diagnostic for dust attenuation/extinction in galaxies and star-forming regions.

Since the intrinsic Hα-to-Hβ ratio can be derived from recombination theory, and is usually assumed to be $(\mathrm{H}\alpha/\mathrm{H}\beta)_0 = 2.87$[1] in the absence of foreground dust, the measured Hα/Hβ measures dust reddening along the line of sight. Since dust affects short wavelengths more than long wavelengths (i.e. affects Hβ at 4861 Å more than Hα at 6563 Å), the Hα/Hβ flux ratio gets boosted by foreground dust.

The relation between observed flux $f(\lambda)$ and intrinsic, unattenunated flux $f(\lambda)_0$ is given by (eq. 2.16 in Schneider's book):

$$f(\lambda) = f(\lambda)_0 \exp(-\tau_\lambda) \tag{7}$$

This may be rewritten (eq. 2.17 in Schneiders book)

$$f(\lambda) = f(\lambda)_0 \times 10^{-0.4A(\lambda)} \tag{8}$$

where $A(\lambda)$ is the dust attenuation (in magnitudes) at wavelength λ.

Adopting $A(\lambda) = k(\lambda)E(B-V)$, where $k(\lambda)$ is an extinction or attenuation law which may differ depending on environment, geometry etc., we can rewrite this as:

$$f(\lambda) = f(\lambda)_0 \times 10^{-0.4k(\lambda)E(B-V)} \tag{9}$$

The attenuation law changes with the properties (size, composition) of the dust grains, the dust-star-gas geometry, and many different attenuation laws are consequently used in the analysis of galaxies. Here, we will use the Calzetti (2000, ApJ, 533, 682) curve, given by:

$$k(\lambda) = 2.659(-1.857 + 1.040/\lambda) + R_V \tag{10}$$

at wavelengths $\lambda = 0.63$–2.2 μm, and

$$k(\lambda) = 2.659(-2.156 + 1.509/\lambda - 0.198/\lambda^2 + 0.011/\lambda^3) + R_V \tag{11}$$

[1]formally valid for a 10,000 K nebula subject to case B recombination

at $\lambda = 0.12$–0.63 μm. Here, R_V is a parameter that relates the V-band dust attenuation A_V to the $E(B-V)$ colour excess due to dust: $R_V = A_V/E(B-V)$. Calzetti et al. (2000) argue for $R_V = 4.05$, which we will use in the following.

Using this machinery, we can derive the ratio of observed-to-intrinsic flux at any wavelength λ (within the range for which the attenuation law is valid; 0.12–2.2 μm in this case), once we have figured out the reddening $E(B-V)$.

Here, we will derive $E(B-V)$ from the ratio of the Balmer lines H_α and H_β. There is a neat little formula that does this (Calzetti et al. 1994, ApJ 429, 582):

$$E(B-V) \approx 0.935 \ln\left(\frac{(\mathrm{H}\alpha/\mathrm{H}\beta)_{\mathrm{obs}}}{(\mathrm{H}\alpha/\mathrm{H}\beta)_0}\right) \tag{12}$$

Once you know $E(B-V)$, you can apply eqs.(9–11) to get the R-band dust attenuation A_R and correct your observed apparent magnitude using:

$$m_{R,0} = m_{R,\mathrm{obs}} - A_R \tag{13}$$

to get the dust-corrected apparent magnitude $m_{R,0}$. Getting the dust-corrected $f_{\mathrm{H}\alpha}$ flux is even easier – you just apply eq.(9):

$$f_{\mathrm{H}\alpha,0} = f_{\mathrm{H}\alpha} \times 10^{0.4k(\lambda_{\mathrm{H}\alpha})E(B-V)} \tag{14}$$

Plugging in the numbers:
Using the observed emission-line ratio of $(\mathrm{H}\alpha/\mathrm{H}\beta)_{\mathrm{obs}} = 3.5$ and an assumed intrinsic ratio of $(\mathrm{H}\alpha/\mathrm{H}\beta)_0 = 2.87$, eq.(12) gives:

$$E(B-V) \approx 0.935 \ln\left(\frac{3.5}{2.87}\right) \approx 0.1856. \tag{15}$$

The R-band has an effective central wavelength of $\lambda \approx 6580$ Å (i.e. 0.6580 μm). At this wavelength, the Calzetti curve (eq. 10) gives:

$$k(\lambda) = 2.659(-1.857 + 1.040/0.6580) + 4.05 \approx 3.31. \tag{16}$$

Plugging $E(B-V) \approx 0.1856$ and $k(\lambda) \approx 3.31$ into $A(\lambda) = k(\lambda)E(B-V)$ gives: $A_R = 0.1856 \times 3.31 \approx 0.61$ mag. With an observed R-band magnitude of $m_R = 23.0$, the dust-corrected R-band magnitude eq.(13) gives:

$$m_{R,0} = m_{R,\mathrm{obs}} - A_R = 23.0 - 0.61 \approx 22.39. \tag{17}$$

The Hα line has a wavelength of $\lambda \approx 6563$ Å (i.e. 0.6563 μm). At this wavelength, the Calzetti curve (eq. 10) gives:

$$k(\lambda) = 2.659(-1.857 + 1.040/0.6563) + 4.05 \approx 3.33. \tag{18}$$

The reason why $k(\lambda)$ end up being so similar in eqs.(16) and (18) is of course that the wavelengths are very close in these two cases. At other wavelengths, the dust corrections will differ greatly from the values derived here. Anyway, adopting $f_{\mathrm{H}\alpha} = 1.0 \times 10^{-15}$ erg s^{-1} cm^{-2}, $k(\lambda) \approx 3.33$ and $E(B-V) \approx 0.1856$ in eq.(9) gives:

$$f_{\mathrm{H}\alpha,0} = 1 \times 10^{-15} \times 10^{0.4\times3.33\times0.1856} \approx 1.8 \times 10^{-15}\ \mathrm{erg\ s^{-1}\ cm^{-2}}. \tag{19}$$

3. *Star formation history*

A local irregular galaxy with a gas mass of $3.8 \times 10^9\ M_\odot$ exhibits very strong Hα emission, $L_{\mathrm{H}\alpha} = 6.0 \times 10^{42}$ erg s^{-1}, indicating ferocious star formation. Assuming a standard Salpeter IMF, the current star formation rate (SFR) may be derived using:

$$\mathrm{SFR}(M_\odot\ \mathrm{yr}^{-1}) = 7.9 \times 10^{-42} L_{\mathrm{H}\alpha}(\mathrm{erg\ s}^{-1}). \tag{20}$$

For how long can this starburst be sustained, assuming:

a) the SFR to be constant over time?

b) the SFR to decrease with time, t, according to $\mathrm{SFR}(t) \propto \exp(-t/\tau)$, where the e-folding decay rate is $\tau = 100$ Myr?

Solution: The gas mass converted into stars over the star formation history of the galaxy is given by:

$$M_{\mathrm{gas}} = \int_0^{t_{\mathrm{max}}} \mathrm{SFR}(t)\ \mathrm{d}t. \tag{21}$$

In the case of constant star formation (case a), the SFR is simply given by:

$$\mathrm{SFR}(t) = \mathrm{SFR}(0). \tag{22}$$

Inserting eq.(22) into eq.(21) implies:

$$M_{\mathrm{gas}} = t_{\mathrm{max}} \mathrm{SFR}(0). \tag{23}$$

Rewriting this gives:

$$t_{\mathrm{max}} = \frac{M_{\mathrm{gas}}}{\mathrm{SFR}(0)}. \tag{24}$$

Inserting eq.(20) into eq.(24) gives:

$$t_{\mathrm{max}} = \frac{M_{\mathrm{gas}}}{7.9 \times 10^{-42} L_{\mathrm{H}\alpha}}. \tag{25}$$

In the case of exponentially declining star formation (case b), the SFR is given by:

$$\mathrm{SFR}(t) = \mathrm{SFR}(0) \exp(-\frac{t}{\tau}) \tag{26}$$

Inserting eq.(26) into eq.(21) implies:

$$M_{\mathrm{gas}} = \int_0^{t_{\mathrm{max}}} \mathrm{SFR}(0) \exp(-\frac{t}{\tau})\ \mathrm{d}t \tag{27}$$

Solving this integral results in:

$$M_{\mathrm{gas}} = \mathrm{SFR}(0)\tau \left(1 - \exp(-\frac{t_{\mathrm{max}}}{\tau})\right) \tag{28}$$

Rearranging gives:

$$t_{\mathrm{max}} = -\tau \ln \left(1 - \frac{M_{\mathrm{gas}}}{\mathrm{SFR}(0)\tau}\right) \tag{29}$$

Inserting eq.(20) into eq.(29) gives:

$$t_{\mathrm{max}} = -\tau \ln \left(1 - \frac{M_{\mathrm{gas}}}{7.9 \times 10^{-42} L_{\mathrm{H}\alpha}\tau}\right) \tag{30}$$

Plugging in the numbers:

Inserting $M_{\mathrm{gas}} = 3.8 \times 10^9\ M_\odot$ and $L_{\mathrm{H}\alpha} = 6.0 \times 10^{42}$ erg s^{-1} into eq.(25) gives:

$$t_{\mathrm{max}} = \frac{3.8 \times 10^9\ M_\odot}{7.9 \times 10^{-42} \times 6.0 \times 10^{42}} \approx 80 \times 10^6\ \mathrm{yr} \tag{31}$$

Hence, star formation can only be sustained for 80 Myr in the case of a constant SFR.

Inserting $M_{\rm gas} = 3.8 \times 10^9\ M_\odot$ and $L_{\rm H\alpha} = 6.0 \times 10^{42}$ erg s^{-1} and $\tau = 100$ Myr into eq.(32) gives:

$$t_{\rm max} = -100 \times 10^6 \ln\left(1 - \frac{3.8 \times 10^9}{7.9 \times 10^{-42} \times 6.0 \times 10^{42} \times 100 \times 10^6}\right) \approx 160 \times 10^6 \text{ yr} \tag{32}$$

Hence, star formation can be sustained for twice as long in the case where the SFR is exponentially declining with an e-folding decay rate of $\tau = 100$ Myr, compared to the constant-SFR scenario.

4. *Emission-line equivalent widths*
The total luminosity of the Hα emission line for a relatively young extragalactic star cluster is inferred to be $L_{\mathrm{H}\alpha} = 4.8 \times 10^{40}$ erg s^{-1}. The monochromatic continuum luminosity right next to the line is $L_{\lambda,\mathrm{H}\alpha\ \mathrm{cont}} = 5 \times 10^{37}$ erg s^{-1} Å^{-1}. Use these data to estimate the Hα equivalent width EW(Hα) of this line and estimte the age of the system.

Solution: The equivalent width EW of an emission line is simply given by the total luminosity in the line (i.e. the monochromatic luminosity integrated over the line profile) divided by the monochromatic continuum luminosity close to the line:

$$\mathrm{EW} = \frac{L_{\mathrm{line}}}{L_{\lambda,\mathrm{cont}}} \quad \frac{[\mathrm{erg\ s^{-1}}]}{[\mathrm{erg\ s^{-1}Å^{-1}}]}. \tag{33}$$

In real-life situations, great care must be exercised when deriving L_{line} and $L_{\lambda,\mathrm{cont}}$ from observed spectra, since:

- The emission line may be sitting in the middle of an absorption feature, which one must then correct for
- The emission line may be blended with nearby lines
- The emission line may have a complicated spectral profile
- The continuum level may be difficult to determine due to a strong continuum slope and/or absorption features

To illustrate the general principle, we will here just ignore such complications and directly apply eq.(33) to derive the equivalent width of the Hα line. Once we have figured out EW(Hα), spectral synthesis models can be used to convert this into an age estimate. Figure 2 illustrates the evolution of EW(Hα) as predicted by the *Yggdrasil* model (Zackrisson et al. 2011, ApJ, 740. 13).

As seen in Figure 2, EW(Hα) decreases with the age of the population. What is the physics behind this? The Hα emission line itself is a recombination line that forms in the photoionized gas surrounding young, hot stars. However, such stars are short-lived, and as soon as they start to die off, the Hα luminosity drops. The continuum beneath the line has two different components:

1. nebular continuum from the photoionized gas. At these wavelengths, it's main free-bound processes; at longer wavelengths free-free emission starts to become more important
2. continuum radiation from stars.

The nebular continuum weakens with age of the stellar population due to the same mechanism that is causing the Hα line luminosity to decrease with age. The stellar continuum, on the other hand, has a more complicated time dependence, and can – for a brief period – actually become stronger over time as red, evolved stars starts to form and contribute to the continuum at these wavelengths.

The combination of these effects produces an EW(Hα) that decreases quickly with age for a single-age stellar population[2]. In the case of more continuous star formation, the reservoir of hot young stars is continuously replenished, but the increasing fraction of evolved, red stars still makes EW(Hα) decrease over time, albeit at a somewhat slower rate. To illustrate this, Figure 2 features two lines – one for a single-age stellar population (red) and one for a population with constant star formation rate (black).

Plugging in the numbers:
Inserting $L_{\mathrm{H}\alpha} = 4.8 \times 10^{40}$ erg s^{-1} and $L_{\lambda,\mathrm{H}\alpha\ \mathrm{cont}} = 5 \times 10^{37}$ erg s^{-1} Å^{-1} into eq.(33) gives:

$$\mathrm{EW(H\alpha)} = \frac{4.8 \times 10^{40}}{5 \times 10^{37}} = 960\ \text{Å}. \tag{34}$$

[2]in the literature, single-age populations are also referred to instantaneous-burst populations, single stellar populations or simple stellar populations

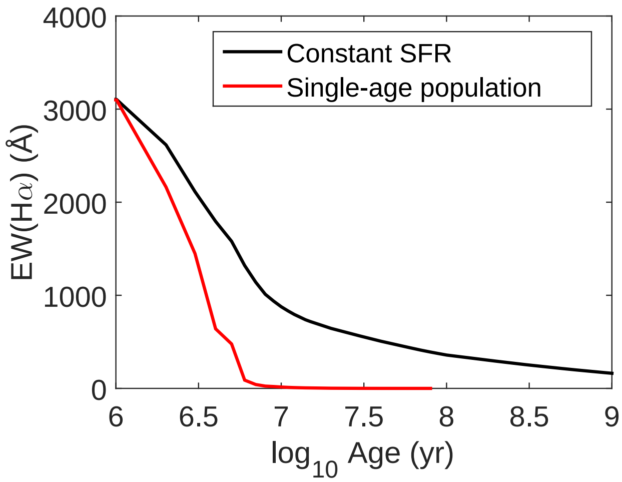

Figure 2: The predicted temporal evolution of the Hα emission-line equivalent width EW(Hα). These models assume a Salpeter IMF and metallicity $Z = 0.020$. The red line represents a single-age stellar population (red line), whereas the black line is for a stellar population that continuously forms new stars with a constant SFR.

Individual star clusters are usually treated as single-age stellar populations, so the model to compare to is the red line in Figure 2. As seen, an Hα equivalent width of EW(Hα)=960 Å is reached at an age of $\log_{10}$ Age $\approx$ 6.5–6.6, i.e. Age $\approx$ 3–4 Myr.

Should EW(Hα) be dust corrected before the comparison to models is made? This is where it gets complicated. The naive answer would be "no", since the emission line and the underlying continuum are sampled at nearly the same wavelength, so even though dust may well have attenuated both the line and the continuum, the amount of attenuation should be that the same for the two components, thereby leaving the equivalent width unaltered. However, in real life, the nebular and stellar contributions may come from different regions with different dust content. It is, for instance, often argued that the Hα line emission in galaxies comes from star-forming regions with a high dust content, whereas the Hα continuum is dominated by old stars, located in regions with low dust content. In this case, dust will decrease EW(Hα), and various correction schemes for this have been developed in the literature. If this mechanism would be at work in our target star cluster, the age estimate we arrived at above would be an upper limit, since the intrinsic (i.e. dust-corrected) EW(Hα) would be higher than the one we derived.

5. *Black holes in galaxies*
A quasar radiating at the Eddington luminosity has a supermassive black hole of mass $4 \times 10^8\ M_\odot$ and is situated in a galaxy with a total gas mass of $1 \times 10^9\ M_\odot$. Use this information to esimate the maximum lifetime of the quasar.

Solution: The Eddington luminosity represents the maximum luminosity that an object powered by the spherical accretion of gas can attain. This is given by:

$$L_{\rm Edd} \approx 1.3 \times 10^{31} \left(\frac{M}{M_\odot}\right)\ [W] \tag{35}$$

where M is the mass of this object. For an accretion rate $\dot{M}$, the luminosity that can be extracted from the accetion process is:

$$L = \eta \dot{M} {\rm c}^2, \tag{36}$$

where η is the efficiency of the energy conversion. Under the assumption of constant accretion rate, the gas mass $M_{\rm gas}$ consumed over a time interval Δt is given by:

$$M_{\rm gas} = \dot{M} \Delta t \tag{37}$$

Rewriting eq.(37) gives:

$$\dot{M} = \frac{M_{\rm gas}}{\Delta t}, \tag{38}$$

and inserting this into eq.(36) gives:

$$L = \frac{\eta M_{\rm gas} {\rm c}^2}{\Delta t}. \tag{39}$$

If the object is radiating at the Eddington luminosity, we can equate (35) and (39) to get:

$$\Delta t = \frac{\eta M_{\rm gas} {\rm c}^2}{1.3 \times 10^{31} (M/M_\odot)}. \tag{40}$$

Plugging in the numbers:
Maximizing the efficiency (i.e. setting $\eta = 1.0$) implies that less gas need to be consumed to sustain the observed luminosity – this sets the maximum lifetime of the quasar. Inserting $\eta = 1.0$, $M = 4 \times 10^8\ M_\odot$ and $M_{\rm gas} = 1 \times 10^9\ M_\odot$ into eq.(40) gives:

$$\Delta t = \frac{1.0 \times 1 \times 10^9\ M_\odot \times 1.99 \times 10^{30}\ {\rm kg} \times \left(3 \times 10^8\ {\rm m\ s^{-1}}\right)^2}{1.3 \times 10^{31} \times 4 \times 10^8\ M_\odot} \approx 3.44 \times 10^{16}\ {\rm s} \approx 1.09 \times 10^9\ {\rm yr}. \tag{41}$$

Hence, the quasar could *in principle* last 1 Gyr, but the efficiency of black hole accretion is typically estimated to be $\eta \approx 0.1$ (slightly less for a non-rotating, Schwarzschild black hole and slightly more for a rotating Kerr black hole). Hence, the maximum lifetime is more likely to be ~ 100 Myr.

Erik Zackrisson, March 2018