Physics of Galaxies 2020

Exercises with solutions – batch I

1. *Distance and brightness at low redshift*
You discover an interesting galaxy in the local Universe and measure its redshift to be $z \approx 0.053$ and its apparent V-band magnitude to be $m_V \approx 25.2$. Use this information to estimate its V-band luminosity. In what way would your estimate change if you found out that this object is part of a galaxy group with a peculiar velocity component of 4000 km s^{-1} directed towards us?

Solution:
To convert the apparent brightness (flux) into a measure of absolute brightness (luminosity), you need to estimate the distance. This holds true at both low and high redshift (local and distant Universe), but the difficulties you face in the two cases are quite different. At high redshifts, the conversion depends on the cosmological model (beyond just knowing the local expansion rate of the Universe, as described by the Hubble parameter H_0), whereas at low redshifts ($z < 0.1$, let's say), you can simply apply Hubble's law. However, the redshift of a nearby object may not be solely due to the expansion of the Univere, since peculiar velocities (movements with respect to the so-called Hubble flow) may easily reach a few times 1000 km s^{-1} and have non-negligible effects on the measured redshift. At higher redshifts, the relative impact of these peculiar velocities become so small that they can (in most cases) be safely ignored.

Estimating the distance:

Hubble's law (eqs. 1.2 to 1.6 in Schneider's book):

$$v = H_0 D, \tag{1}$$

where v is the so-called recession velocity and D is the distance. At low redshifts, the recession velocity is related to the redshift through:

$$v = cz, \tag{2}$$

where c represents the speed of light. To convert redshift into a distance estimate using Hubble's law, you simply insert eq.(2) into eq.(1) and rearrange:

$$D = \frac{cz}{H_0}. \tag{3}$$

In the case where part of the redshift comes from a peculiar motion you need to split the recession velocity in eq.(2) into two components v_{peculiar} and v_{Hubble}:

$$v_{\text{peculiar}} + v_{\text{Hubble}} = cz. \tag{4}$$

Please note that when the peculiar velocity is directed towards you (thereby effectively causing a blueshift), it is considered negative in this notation, whereas is a peculiar velocity directed away from you (causing a redshift) is considered positive. To make sure that the velocity you use when applying Hubble's law to get the distance only reflects the explansion of the Universe (and not the peculiar motion of the galaxy), you should rearrange eq.(4) so that $v_{\text{Hubble}} = cz - v_{\text{peculiar}}$ and insert this into eq.(1). After rearranging once more, you get:

$$D = \frac{cz - v_{\text{peculiar}}}{H_0}. \tag{5}$$

Estimating the luminosity:
The luminosity in the V-band (rest-frame wavelength range $\approx$ 4800–6500 Å) can be expressed either as an absolute V-band magnitude M_V, or as a luminosity L_V in absolute terms (either in units of $L_{\odot,V}$ or in erg s^{-1}, J s^{-1} or W).

To get the absolute V-band magnitude, you simply apply:

$$M_V = m_V - 5\log_{10} D_{\rm pc} + 5 - A_V, \tag{6}$$

where $D_{\rm pc}$ is the distance to the galaxy in parsecs and A_V is the V-band dust attenuation along the line of sight. To go from M_V to L_V, you can apply eq. A.34 from Schneider's book:

$$L_V = 10^{\frac{M_V - M_{V,\odot}}{-2.5}}\ L_{V,\odot}, \tag{7}$$

where $M_{V,\odot}$ is the absolute magnitude of the Sun in the V-band and $L_{V,\odot}$ is the V-band luminosity of the Sun. The solar values are often taken to be $M_{V,\odot} \approx 4.84$ and $L_{V,\odot} \approx 4 \times 10^{32}$ erg s^{-1}.

Plugging in the numbers:
Case A: Redshift assumed to be solely due to the expansion of the Universe.
$z \approx 0.053$, $c = 3 \times 10^5$ km s^{-1}, $H_0 \approx 70$ km s^{-1} Mpc^{-1} in eq.(2) $\Rightarrow$

$$D = \frac{cz}{H_0} = \frac{3 \times 10^5\ \times\ 0.053}{70}\ \frac{[\text{km s}^{-1}]}{[\text{km s}^{-1}\text{Mpc}^{-1}]} \approx 227 \text{ Mpc}. \tag{8}$$

$m_V \approx 25.2$, $A_V \approx 0$ (dust neglected) and $D_{\rm pc} \approx 227 \times 10^6$ pc in eq.(6) $\Rightarrow$

$$M_V = m_V - 5\log_{10} D_{\rm pc} + 5 + A_V = 25.2 - 5\log_{10}(227 \times 10^6) + 5 - 0 \approx -11.58. \tag{9}$$

$M_V \approx -11.6$, $M_{V,\odot} \approx 4.84$ in eq.(15) $\Rightarrow$

$$L_V = 10^{\frac{M_V - M_{V,\odot}}{-2.5}}\ L_{V,\odot} = 10^{\frac{-11.58-4.84}{-2.5}}\ L_{V,\odot} \approx 3.7 \times 10^6\ L_{V,\odot} \approx 1.48 \times 10^{39} \text{ erg s}^{-1}. \tag{10}$$

Case B: Redshift partly due to peculiar velocity, and partly the expansion of the Universe.
$z \approx 0.053$, $v_{\rm peculiar} = -4000$ km s^{-1}, $c = 3 \times 10^5$ km s^{-1}, $H_0 \approx 70$ km s^{-1} Mpc^{-1} in eq.(5) $\Rightarrow$

$$D = \frac{cz - v_{\rm peculiar}}{H_0} = \frac{3 \times 10^5\ \times\ 0.053 - (-4000)}{70}\ \frac{[\text{km s}^{-1}]}{[\text{km s}^{-1}\text{Mpc}^{-1}]} \approx 284 \text{ Mpc}. \tag{11}$$

$m_V \approx 25.2$, $A_V \approx 0$ (dust neglected) and $D_{\rm pc} \approx 284 \times 10^6$ pc in eq.(6) $\Rightarrow$

$$M_V = m_V - 5\log_{10} D_{\rm pc} + 5 + A_V = 25.2 - 5\log_{10}(284 \times 10^6) + 5 - 0 \approx -12.066. \tag{12}$$

$M_V \approx -12.1$, $M_{V,\odot} \approx 4.84$ in eq.(15) $\Rightarrow$

$$L_V = 10^{\frac{M_V - M_{V,\odot}}{-2.5}}\ L_{V,\odot} = 10^{\frac{-12.066-4.84}{-2.5}}\ L_{V,\odot} \approx 5.8 \times 10^6\ L_{V,\odot} \approx 2.32 \times 10^{39} \text{ erg s}^{-1}. \tag{13}$$

Comment: Neither of these estimates take dust attenuation (A_V) into account. The apparent magnitude (flux) of a galaxy may be dimmed by dust both along the line of sight and within the galaxy itself. Depending on you are going to use your luminosity estimate for, you may want to take at least the line-of-sight component into account. Let's say that this particular galaxy happens to be seen along a sightline through the Milky Way that is attenuated by 0.3 magntiudes in the V-band (there are Milky Way dust maps and online tools that can help you figure this out, based on the coordinates of your object). How would this then affect the luminosity estimate? As an example, let's insert $A_V = 0.3$ mag into eq.(12):

$$M_V = m_V - 5\log_{10} D_{\rm pc} + 5 + A_V = 25.2 - 5\log_{10}(284 \times 10^6) + 5 - 0.3 \approx -12.37. \tag{14}$$

and

$$L_V = 10^{\frac{M_V - M_{V,\odot}}{-2.5}}\ L_{V,\odot} = 10^{\frac{-12.37-4.84}{-2.5}}\ L_{V,\odot} \approx 7.7 \times 10^6\ L_{V,\odot} \approx 3.06 \times 10^{39}\ \text{erg s}^{-1}. \tag{15}$$

Hence, our V-band luminosity estimate changed by a factor of $10^{(-0.3/-2.5)} \approx 1.32$ when taking dust effects into account.

2. *Brightness and colours of unresolved stellar systems*
Two stars in a close binary system have $m_{\mathrm{B},1} = 18.2$ and $m_{\mathrm{B},2} = 19.6$, respectively. The first star has a colour $B - V = -0.2$ and the second $B - V = 0.5$. If this system is observed in a telescope which cannot resolve the two components, what would the integrated m_B and $(B - V)$ colour of this object be?

Solution: At large distances, it becomes increasingly difficult to study the individual stars of stellar populations. There are basically two reasons for this:

1. The flux sensitivity of your telescope may be insufficient to allow the detection of a single, very distant star, whereas the combined light from large numbers of stars at the same distance may allow them to be detected

2. Limitations in angular resolution of your telescope may blend the light from stars that are close in the sky to one another, thereby making distant stellar populations appear as single objects.

Much of contemporary extragalactic astronomy is therefore devoted to the study of the *integrated* light from large numbers of stars. To illustrate this point, Figure. 1 shows the appearance of stellar populations at increasing distances, when observed with current telescopes.

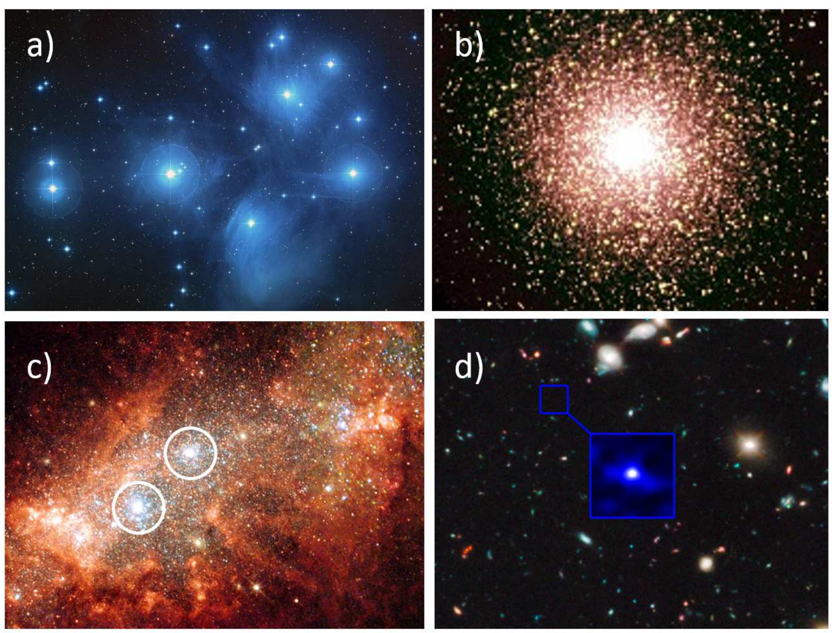

Figure 1: Resolved versus unresolved stellar populations. **a)** A nearby star cluster (the Pleiades, distance 440 ly) in which individual stars can be resolved; **b)** a Milky Way globular cluster (47 Tucanae, distance 17,000 ly) where many stars blur into a single orb of light in the centre; **c)** Two unresolved super star clusters in the local galaxy NGC 1569 (distance 11 Mly); **d)** The unresolved distant galaxy candidate UDFj-39546284 (for a while considered to be the most distant galaxy known, at redshift $z \approx 11.9$, now believed to be a lower-redshift interloper at $z \approx 2$–3). Image credits: (a) NASA, ESA, AURA/Caltech, Palomar Observatory; (b) SALT; (c) ESA, NASA and P. Anders; (d) NASA, ESA, Garth Illingworth, Rychard Bouwens and the HUDF09 Team.

Let's consider the simplest case - where two stars lie so close together in the sky that they cannot be spatially resolved and are therefore seen as a single object in your telescope. What are the expected properties of that object?

To get the total flux of a stellar system, you bascially just sum up the flux from the constituents – in this case the two stars. The only problem is that the fluxes are here given as apparent magnitudes, and magnitudes are logarithmic quantities that cannot be summed up directly. So, what do you do?

The first step is to convert the apparent magnitudes $m_{B,1}$ and $m_{B,2}$ of the two stars into fluxes, $f_{B,1}$ and $f_{B,2}$:

$$f_{B,1} = 10^{\frac{m_{B,1}}{-2.5}} \tag{16}$$

$$f_{B,2} = 10^{\frac{m_{B,2}}{-2.5}} \tag{17}$$

Now, sum these up to get the total B-band flux:

$$f_{B,\mathrm{tot}} = f_{B,1} + f_{B,2} \tag{18}$$

Next, convert the total B-band flux back into an apparent magnitude:

$$m_{B,\mathrm{tot}} = -2.5 \log_{10} f_{B,tot} \tag{19}$$

This represents the total apparent magnitude of the binary star system. If you want a generalized expression, you can combine eqs.(16)–(19) to get the integrated (total) flux:

$$m_{B,\mathrm{tot}} = -2.5 \log_{10} \sum_i 10^{\frac{m_{B,i}}{-2.5}} \tag{20}$$

This actually works for *any* number of stars – you can sum up the individual flux from billions of stars to get the total flux of an entire galaxy.

To get the total $B - V$ colour of the stellar population, you simply subtract the V-band magnitude from the B-band one:

$$(B - V)_{\mathrm{tot}} = m_{B,\mathrm{tot}} - m_{V,\mathrm{tot}}. \tag{21}$$

In the case of this binary system, you derive the V-band magnitudes of the two stars using:

$$V_1 = B_1 - (B - V)_1 \tag{22}$$

and

$$V_2 = B_2 - (B - V)_2. \tag{23}$$

To get the total V-band magnitude $m_{V,\mathrm{tot}}$ of this system, you just replace the Bs in eqs.(16)–(19) by V:

$$m_{V,\mathrm{tot}} = -2.5 \log_{10} \sum_i 10^{\frac{m_{V,i}}{-2.5}} \tag{24}$$

Plugging in the numbers:
$m_{\mathrm{B},1} = 18.2$ and $m_{\mathrm{B},2} = 19.6$ in eq.(20) ⇒

$$m_{B,tot} = -2.5 \log_{10} \left[10^{\frac{18.2}{-2.5}} + 10^{\frac{19.6}{-2.5}} \right] \approx 17.9359 \tag{25}$$

$(B - V)_1 = -0.2$ and $(B - V)_2 = 0.5$ in eq.(22) and eq.(23) ⇒

$$V_1 = 18.2 - (-0.2) = 18.4 \tag{26}$$

$$V_2 = 19.6 - 0.5 = 19.1 \tag{27}$$

Inserting $V_1 = 18.4$ and $V_2 = 19.1$ into eq.(24) ⇒

$$m_{V,tot} = -2.5 \log_{10} \left[10^{\frac{18.4}{-2.5}} + 10^{\frac{19.1}{-2.5}} \right] \approx 17.9420 \tag{28}$$

The total $(B - V)_{\mathrm{tot}}$ colour then becomes (eq.21):

$$(B - V)_{\mathrm{tot}} = m_{B,tot} - m_{V,tot} = 17.9359 - 17.9420 \approx 0.0. \tag{29}$$

Hence, the brighter of the two stars ($m_{B,1} = 18.2$, $(B - V)_1 = -0.2$) has a larger impact on the total colour than the fainter one ($m_{B,2} = 19.6$, $(B - V)_2 = 0.5$), in the sense that the total colour becomes closer to that of the brighter star than the fainter one.

3. *The stellar initial mass function (IMF)*
If a galaxy has a star formation rate of 10 $M_\odot$ yr^{-1}, how many newly formed stars per year does this correspond to, if we assume a Salpeter IMF throughout the interval $0.1 - 120\ M_\odot$?

Solution: The definition of the Salpeter IMF is (eq. 3.36 in Schneider's book):

$$\Phi = \Phi_0 \int_{\mathrm{M_{min}}}^{\mathrm{M_{max}}} M^{-2.35} \mathrm{d}M, \tag{30}$$

where Φ corresponds to the number of stars formed in the mass interval $[M_{min}, M_{max}]$, and Φ_0 is a normalization constant. The problem in this exercise basically amounts to figuring out the value of Φ_0. In general, Φ can either represent the total number of stars, or the total number of stars per unit time. Here, it's more convenient to consider the latter option, with Φ given in units of stars yr^{-1}.

In this exercise, we know what gas mass is converted into stars every year. The expression for this star formation rate (SFR(t); usually given in units of $M_\odot$ yr^{-1}) can be derived by multiplying the number of stars produced in a given mass interval each year by their individual masses (see page 133 in Schneider's book):

$$\mathrm{SFR}(t) = \Phi_0 \int_{\mathrm{M_{min}}}^{\mathrm{M_{max}}} M \times M^{-2.35} \mathrm{d}M = \Phi_0 \int_{\mathrm{M_{min}}}^{\mathrm{M_{max}}} M^{-1.35} \mathrm{d}M. \tag{31}$$

Rewriting eq.(31), we can solve for Φ_0:

$$\Phi_0 = \frac{\mathrm{SFR}(t)}{\int_{\mathrm{M_{min}}}^{\mathrm{M_{max}}} M^{-1.35} \mathrm{d}M}. \tag{32}$$

Eq.(32) may then be substituted into eq.(31) $\Rightarrow$

$$\Phi = \frac{\mathrm{SFR}(t) \int_{\mathrm{M_{min}}}^{\mathrm{M_{max}}} M^{-2.35} \mathrm{d}M}{\int_{\mathrm{M_{min}}}^{\mathrm{M_{max}}} M^{-1.35} \mathrm{d}M} \tag{33}$$

Plugging in the numbers:
If we adopt $M_{\mathrm{min}} = 0.1\ M_\odot$ and $M_{\mathrm{max}} = 120\ M_\odot$, and use star formation rate SFR(t) = 10 $M_\odot$ yr^{-1} in eq.(33) $\Rightarrow$

$$\Phi = \frac{10\ M_\odot \mathrm{yr}^{-1} \int_{0.1\ M_\odot}^{120\ \mathrm{M}_\odot} M^{-2.35} \mathrm{d}M}{\int_{0.1\ M_\odot}^{120\ M_\odot} M^{-1.35} \mathrm{d}M} \tag{34}$$

Let's solve the integrals $\Rightarrow$

$$\Phi = \frac{-0.35 \times 10\ M_\odot \mathrm{yr}^{-1} \left[120^{-1.35} - 0.1^{-1.35}\right]}{-1.35 \times \left[120^{-0.35} - 0.1^{-0.35}\right]} \approx 28 \text{ stars yr}^{-1} \tag{35}$$

4. *Rotation curves*

An spiral galaxy displays the inclination-corrected rotation curve shown in Figure 2. Use this rotation curve to set a lower limit on the mass of this system.

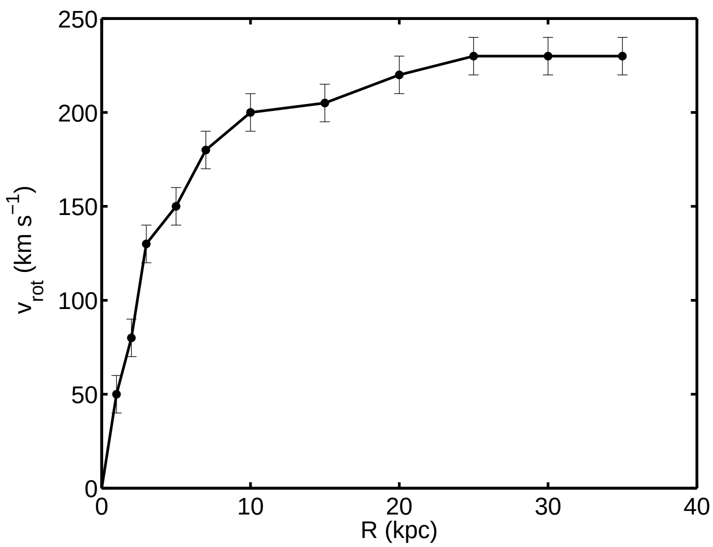

Figure 2: The inclination-corrected rotation curve of a disk galaxy (rotational velocity as a function of distance from the centre of the disk).

Solution: Consider a test particle of mass m orbiting an object of much higher mass M. By measuring the distance r between these bodies, and the velocity v of the test particle, it is possible to infer the mass M of the central object.

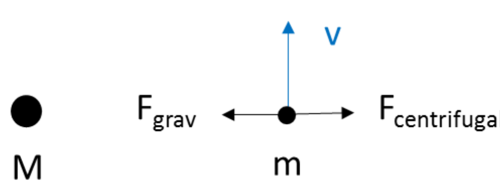

Figure 3: Schematic illustration of a test particle of negligible mass m with velocity v, in circular orbit around more massive object with mass M. The gravitational force $F_{\rm grav}$ is here equal to the centrifugal (pseudo-)force $F_{\rm centrifugal}$.

How does this work? Well, if the gravitational and centrifugal forces are to cancel each other, we have

$$\frac{GmM}{r^2} = \frac{mv^2}{r}, \tag{36}$$

where G is the gravitational constant. If we rearrange, we find that m cancels out, and we get:

$$M = \frac{v^2 r}{G} \tag{37}$$

Hence, by measuring v and r, we can infer M. If we generalize the simple situation of a particle of negligible mass m orbiting a more massive object M into a tracer particle moving in a spherical dark matter halo, the mass M in eq.(37) is usually interpreted as the mass $M(< r)$, i.e. the mass enclosed with the radius r. By measuring the rotational velocity of some tracer particles (stars, or gas clouds) at different distances from the centre, we can derive the mass of the galaxy enclosed within that radius. Now, detectable tracer particles tend to be concentrated towards the inner region of the dark matter halo (where the luminous galaxy is sitting), so observed rotation curves never extend to the outer limit of the dark halo. For this reason, the mass you derive using eq.(37) will in practice be a lower limit to the total mass – if you could measure velocities even further out, the mass estimate would likely increase somewhat. If you want to impose the strongest possible limit on the mass of the galaxy which displays a rotation curve that stays flat out to the outermost measured data point, it's a good idea to use that outermost datapoint for your calculations.

Plugging in the numbers:

Let's pick the outermost datapoint (R = 35 kpc, v = 230 km s^{-1}) from the rotation curve in Fig. 2 and apply eq.(37) – the main problem here is to keep track of the units. A parsec (pc) is about 3.0857×10^{16} m, the gravitational constant G is 6.67384×10^{-11} m^3 kg^{-1} s^{-2}in SI units, and the mass of the Sun is 1.989×10^{30} kg.

$$M = \frac{(230 \times 10^3)^2 \times 35 \times 10^3 \times 3.0857 \times 10^{16}}{6.67384 \times 10^{-11}} \approx 8.56 \times 10^{41} \text{ kg} \tag{38}$$

Converting this into Solar masses yields $4.3 \times 10^{11}\ M_\odot$.

Comment: There are quite a few assumptions going into this derivation. To begin with, the mass of the galaxy is assumed to be centrally concentrated (similar to how the Sun makes up most of the mass of the the Solar System), but this is simply not so. The modern view of a disk galaxy includes a disk of stars and gas, a stellar bulge, a stellar halo, a dark matter halo and potentially a gaseous, highly ionized halo component. Because of the different geometrical shapes of these components, relating rotational velocity to mass is in general highly non-trivial. However, in the simplifying case where a spherical dark matter halo is assumed to dominate the mass budget at all radii, the simple Newtonian analysis presented here (and in Schneider's book) is usually considered adequate.

5. *The Tully-Fisher relation.* An edge-on disk galaxy is observed to have the HI line profile depicted in Fig. 4 and an H-band flux of $m_{\rm H} = 15.1$. Use the Tully-Fisher relation to estimate the distance to this object.

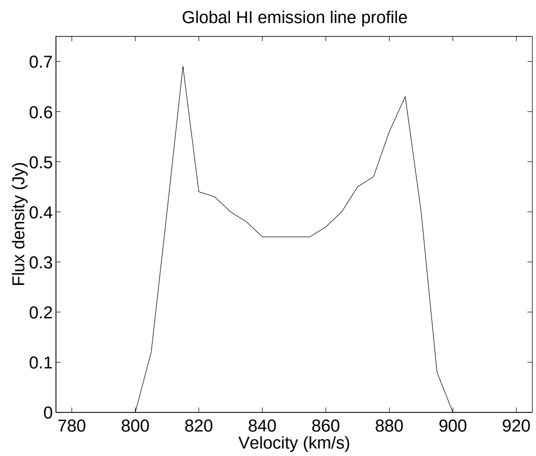

Figure 4: The global HI profile of an edge-on disk galaxy. While a light spectrum usually features wavelength or frequency on the x-axis, this quantity has here for been converted into a velocity to illustrate the redshifts/blueshifts that HI clouds at different positions along the rotating disk exhibit compared around the so-called systemic velocity (which is produced by the expansion of the Universe and the peculiar motion of the galaxy relative to the Hubble flow).

Solution: The HI line profile of disk galaxies contains information about the velocity distribution of gas clouds within these systems. The line profile encodes information somewhat similiar to that in a rotation curve (e.g. Fig. 2), but doesn't require the galaxy to be spatially resolved and can therefore be observed for more distant systems. Basically, the HI profile tells you what the typical velocity of the HI reservoir in the galaxy is. Since most of the HI is in the outskirts, where the rotational velocity has plateaued, the line width can be used to estimate the maximal velocity of the rotation curve. As discussed in exercise 4, this is related to the overall mass of the system.

Empirically, the HI line width is tightly coupled to the absolute magnitude. This is illustrated for observations in various filters in Fig. 3.27 in Schneider's book. So, if you know the HI line width W, you can predict the absolute magnitude M. If you couple this to a measurent of the apparent magnitude m, you can derive the distance to the galaxy using the standard relation

$$M = m - 5\log_{10} D_{\rm pc} + 5 - A, \tag{39}$$

where $D_{\rm pc}$ is the distance in parsec and A is the dust attenuation along the line of sight.

To convert the measured HI line width W into an absolute magnitude in the H-band, you may either read off M_H directly from Fig. 3.27 in Schneider's book, or use the empirical relation on which that plot has been based (Pierce & Tully 1992, ApJ, 387, 7):

$$M_H \approx -9.50(\log_{10} W - 2.50) - 21.67, \tag{40}$$

where W is in units of km s^{-1}.

Since we are dealing with flux measurements in the infrared, where the effects of dust attenuation are typically very small, we neglect the attenuation correction A in eq.(39) and solve for the distance:

$$D_{\rm pc} = 10^{\frac{M_H - m_H - 5}{-5}}. \tag{41}$$

Plugging in the numbers:
By measuring $W \approx 68$ km s^{-1} from Fig. 4 and inserting this into eq.(42), we get:

$$M_H \approx -9.50(\log_{10} 68 - 2.50) - 21.67 \approx -15.33 \tag{42}$$

Inserting $M_H = -15.33$ and $m_H = 15.1$ into eq.(41) $\Rightarrow$

$$D_{\rm pc} = 10^{\frac{-15.33-15.1-5}{-5}} \approx 12.2 \text{ Mpc} \tag{43}$$

Erik Zackrisson, March 2020