# Cosmology AS7009

# 2009

## Exercises to be solved in class

**1.** *Olbers' paradox: Why is the sky dark at night?* Let's assume that the universe is static and of infinite extent. The number density of stars is $n_\star$, the average stellar luminosity $L_\star$ and the average area of a stellar disc $A_\star$.

a) What is the total flux observed at some arbitrary point in the Universe?

b) If you furthermore assume that each star is covering an angle $A_\star/r^2$ on the sky and effectively hides the light from an infinite number of more distant objects, it may no longer be appropriate to assume the line of sight to go to infinity. Instead, one should integrate the flux received from stars only out to a distance $D_{\rm max}$, for which the whole sky will be evenly tiled by stellar discs. Derive this distance $D_{\rm max}$ and calculate the integrated flux from all the visible stars out to that distance. Does this resolve the paradox?

c) Describe at least four possible ways to solve Olbers' paradox.

**2.** *Defining the Universe.* What exactly do we mean by "the Universe"?

a) Suggest a suitable definition. What should and shouldn't the concept of a Universe encompass?

b) Is there any difference between the Universe and the observable Universe?

c) Can there be more than one Universe?

**3.** *Fun with the fluid equation: density evolution.* Use the fluid equation $\dot\rho c^2 + 3\frac{\dot a}{a}(\rho c^2 + p) = 0$ and the equation of state $p = w\rho c^2$ to derive a proportionality relation between density and scale factor in the case of

a) a radiation dominated Universe

b) a matter dominated Universe

c) a Universe dominated by a cosmological constant.

**4.** *Fun with the Friedmann equation: the age of the Universe.* Starting from the Friedmann equation, derive an expression for the age of the Universe $t(z)$, in the case of a standard cosmological model including the parameters $\Omega_{\rm rad}$, $\Omega_{\rm M}$, $\Omega_\Lambda$, $\Omega_{\rm tot}$.

a) What does the relation for $t(z)$ reduce to in the case of an empty (Milne) Universe?

b) What does the relation for $t(z)$ reduce to in the case of an Einstein-de Sitter Universe?

c) What is the current age of the Universe if $\Omega_{\rm rad} = 8.4\times10^{-5}$, $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$, $\Omega_{\rm tot} \approx 1.0$? What was it at redshift $z = 1$?

**5.** *Is space infinite?*

a) How large is the observable Universe in the standard Big Bang scenario?

b) Is the Universe larger than the part that we can in principle observe? What is going on outside the horizon? Is space outside infinite?

c) If the universe is infinite now – how large was it at the time of the Big Bang?

d) Could the universe in principle be smaller than the region defined by the horizon?

**6.** *Cosmological distances.* Consider a light source at a redshift of $z = 3$ in an Einstein-de Sitter Universe.

a) How far has the light from this object traveled to reach us?
b) How distant is this object today?
c) How distant was this object at the epoch when the light was emitted?
d) How far would light emitted from us today have to travel to reach the object?

**7.** *Standard candles.* Estimate the expected apparent magnitude $m_{\rm B}$ of a supernova type Ia (absolute magnitude $M_{\rm B} \approx -19.6$) at a redshift of $z = 0.8$ in a Universe described by $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$, $H_0 = 72$ km s$^{-1}$ Mpc$^{-1}$. Cosmological $k$-corrections and dust effects may be neglected.

**8.** *Matter-radiation equality.* Estimate the redshift and temperature of the photon gas at the transition from radiation to matter domination, assuming a $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0$ cosmology.

**9.** *Dark energy and the big rip.* If the dark energy has an equation of state $w < -1$ , the Universe may be ripped apart as the scale factor $a \to \infty$ when $t \to t_{\rm Rip}$. Derive an analytical expression for $t_{\rm Rip}$, under the assumption that the Universe has a flat geometry and is currently dominated by dark energy with constant $w$. Predict the time remaining before the Big Rip, in scenarios where:

a) $w = -1.1$
b) $w = -1.5$
c) $w = -2.0$

**10.** *Dark matter I.* Some unknown scientist launches a new theory about the nature of dark matter, in which the dark matter mainly consists of self-reproducing robots designed several billion years ago by some advanced civilization eager to explore the Universe. Ever since the demise of their masters in an unforeseen supernova explosion, the robots have been on a cosmic rampage, turning anything they can get their hands on into new robots. The founder of this theory estimates that essentially all the matter of the universe should now be in the form of such robot probes. Apparently, he has also located a broken specimen of this robot population buried in his own backyard. He is however not prepared to show this piece of alien technology to the public just yet. At a conference, he is confronted with the question: "But what are these robots made of?"

He thinks about this for a long time and then replies:

"Metal... Yeah, and some plastic."

Try to estimate an upper limit to the contribution of self-reproducing robots, $\Omega_{\rm robots}$ to the cosmological density of the Universe, assuming the standard Big Bang scenario to be correct.

**11.** *Dark matter II.* An estimated upper limit to the mass of the neutrino is 10 eV. Would this be enough to explain the estimated amount of dark matter in the Universe? What would the upper limit to the neutrino mass be if $\Omega_{\rm M} = 0.3$?

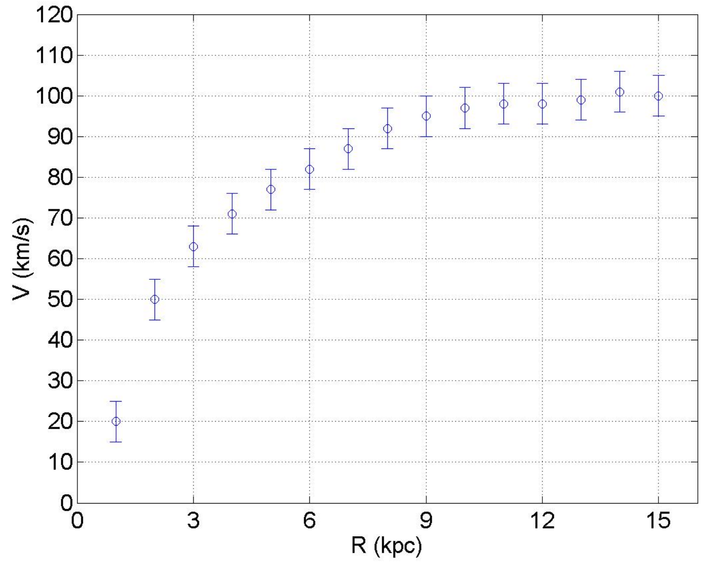


Figure 1: The rotation curve of a disk galaxy, projected to edge-on orientation.

**12.** *Dark matter III* A disk galaxy at a redshift of $z = 0.0020$ displays an apparent magnitude of $m_{\rm B} = 11.1$ and the inclination-corrected rotation curve shown in Fig. 1. Use this data to set a lower limit to the mass and mass-to-light ratio $M/L_{\rm B}$ (in units of $\frac{M_\odot}{L_{\rm B,\odot}}$) of this system, and a lower limit to the contribution from dark matter to the total mass, given that models indicate that the mass-to-light ratio of the visible component should be $M/L_{\rm B} \approx 2$. You may find it useful to know that $M_{B,\odot} = 5.48$.

**13.** *Gravitational lensing*
Dark halo substructure can in principle be detected by its gravitational lensing effects (so-called meso- or millilensing) on background light sources. If the substructures are approximated to be point-masses of mass $M_{\rm sub}$, the image-splitting angle can be estimated from the angular Einstein radius

$$\theta_{\rm E} = \left(\frac{4GM_{\rm sub}D_{\rm ls}}{c^2 D_{\rm ol} D_{\rm os}}\right)^{1/2}, \quad (1)$$

where $D_{\rm ls}$, $D_{\rm ol}$ and $D_{\rm os}$ are the angular size distances between lens and light source, observer and lens, observer and light source, respectively. Calculate the image separation angle (in arcseconds) for a light source at $z = 2$, produced by a dark subhalo of mass $M = 10^7\ M_\odot$ located at

a) a redshift of $z = 0.5$
b) a redshift of $z = 1.0$
c) a distance of 100 kpc (Note: This is inside the dark halo of the Milky Way!)

**14.** *Age, redshift and temperature at recombination.* Estimate the redshift of recombination, assuming this event to take place when the temperature of the photon gas has dropped to 4000 K. What age of the universe does this correspond to, assuming an Einstein-de Sitter cosmology?

**15.** *Cosmic Microwave Background Radiation (CMBR) I.* The relation of proportionality between the energy density and the temperature of the CMBR is $\rho_{\rm CMBR} \propto T^4_{\rm CMBR}$. Use this information to derive the number density of CMBR photons as a function of redshift. What was the number density at $z = 5$?

**16.** *Big bang nucleosynthesis.* When the reactions that convert protons into neutrons fall out of equilibrium ($t_{\rm universe} \sim 1$ s old), $\frac{n_n}{n_p} \approx 0.18$. Some of the neutrons will however decay back into protons before the formation of $^4$He. Estimate the mass fraction of $^4$He formed if this is assumed to take place when $t_{\rm universe} \approx 180$ s.

**17.** *The flatness problem I.* Use the Friedmann equation

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} - \frac{\kappa c^2}{R_0^2 a^2}$$

to derive

$$\Omega_{\rm tot}(t) - 1 \propto \frac{1}{a(t)^2 H(t)^2}$$

**18.** *Size-redshift relation.* For flat cosmologies, the angular size $\Theta$ of an object with intrinsic linear size $l$ (perpendicular to the line of sight) at redshift $z$ is given by (small $\Theta$)

$$\Theta = \frac{l}{D_{\rm A}}$$

where $D_{\rm A}$ is the angular-size distance:

$$D_{\rm A} = \frac{c}{H_0(1+z)} \int_1^{1+z} \frac{dx}{(\Omega_{\rm M} x^3 + (1 - \Omega_{\rm M} - \Omega_\Lambda)x^2 + \Omega_\Lambda)^{1/2}}.$$

a) Make a plot of $\Theta(z)$ for a $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$ universe (the integration will have to be done numerically, e.g. in Matlab). Comment on your (strange?) results.

b) What is the angular size of a galaxy at $z = 1$ with an intrinsic size of $10^5$ ly? What would the angular size of the same object be if it was located at $z = 10$?

# Hand-in exercises

**1.** *Hubble's law and luminosity distance.* A galaxy is observed at a redshift of $z = 0.25$. How distant is this object according to Hubble's law? How accurate is Hubble's law for estimating the luminosity distance at this redshift, under the assumption of a cosmological model with $\Omega_{\rm M} = 0.3$ and $\Omega_\Lambda = 0.7$?

**2.** *Horizon distance.* In a de Sitter universe, the time dependence of the scale factor is given by $a(t) = \exp \mathrm{H}_0(t - t_0)$.

a) Derive an expression for the horizon distance in this cosmology

b) Is the horizon distance larger or smaller than the Hubble distance in a 13.7 Gyr old de Sitter universe where $\mathrm{H}_0 = 72$ km s$^{-1}$ Mpc$^{-1}$?

**3.** *Fate of the Universe.* Starting from the Friedmann equation

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi G\rho}{3} - \frac{\kappa c^2}{R_0^2 a^2}, \tag{2}$$

demonstrate that a currently expanding, matter-only universe will continue to expand forever if $\Omega_{\rm M} \leq 1$, but not if $\Omega_{\rm M} > 1$.

**4.** *The era of dark-energy domination.* Estimate the redshift at which the Universe became dark-energy dominated, assuming $\Omega_{\rm M} = 0.3$ and $\Omega_{\rm DE} = 0.7$ today, and that the dark energy has an equation of state ($p = wc^2\rho$):

a) $w = -1.0$ (i.e. a cosmological constant)

b) $w = -1.5$

**5.** *Dark energy and supernovae type Ia.* The redshifts and apparent magnitudes of a small sample of supernovae type Ia are listed in Table 1. Use this data to determine which of the following three cosmological models is the most likely:

a) $\Omega_{\rm M} = 1.0$, $\Omega_\Lambda = 0.0$

b) $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$

c) $\Omega_{\rm M} = 0.5$, $\Omega_\Lambda = 1.5$

You can assume $H_0 = 72$ km s$^{-1}$ Mpc$^{-1}$ in all cases. Both dust extinction corrections and cosmological $k$-corrections may be ignored. The estimated photometric $1\sigma$ errors of the measurements are $\sigma_m = 0.3$ magnitudes. Errors in the redshift determinations can be assumed negligible. Note: As this is just a toy example with artificial data, you should not expect the concordance model to win by default.

**6.** *Dark matter IV.* Let's assume the existence of a dark population of primordial black holes (PBHs) with mass $10^{-10}$ $M_\odot$ and cosmological density $\Omega_{\rm PBH} = 0.2$. If these objects are gravitationally clustered to the dark halos of galaxies with a density enhancement factor of $10^5$ relative to the cosmic average, how many such objects would you then expect to find in the vicinity of the solar system, i.e. inside a sphere with radius equal to the mean distance from the Sun to Pluto? How many would you find inside a sphere with the same radius as the Milky Way? You may assume dark matter halos to have constant density.

**7.** *Cosmic Microwave Background Radiation II.* The CMBR can be very well represented by a black-body spectrum with a current temperature of 2.728 K. Use this information to estimate its contribution $\Omega_{\rm CMBR}$ to the cosmological density.

**8.** *The flatness problem II.* If $|\Omega_{\rm tot} - 1| < 0.1$ now, what value of $|\Omega_{\rm tot} - 1|$ does that imply at

a) the epoch of matter-radiation equality?

b) the Planck time, assuming no inflation?

In both-cases, late-time domination by dark energy may be neglected.

**9.** *Common misconceptions about cosmic expansion.* The following statement reveals a common misconception about the time evolution of $H$: "The Hubble parameter measures the expansion rate of the Universe, and since the Universe is accelerating now, $H$ must have been smaller in the past and will in the future become greater than its present value." Explain what is wrong with the reasoning underlying this statement, and derive the true temporal evolution of $H$.

Table 1: Supernova Type Ia data

| $z$ | $m_B$ |
|---|---|
| 0.022 | 15.3 |
| 0.041 | 17.0 |
| 0.057 | 17.4 |
| 0.083 | 18.3 |
| 0.11 | 18.7 |
| 0.35 | 22.0 |
| 0.42 | 22.1 |
| 0.57 | 23.5 |
| 0.83 | 24.9 |
| 0.90 | 24.5 |
| 0.98 | 25.2 |
| 1.10 | 25.6 |
| 1.30 | 25.2 |
| 1.50 | 25.7 |
| 1.9 | 26.9 |

**Preliminary schedule for exercise classes**

Remember that it will be much easier to understand the solutions presented during the exercise sessions if you have already attempted to solve the problems yourself.

Exercise session 1 - Problems 1–6
Exercise session 2 - Problems 7–12
Exercise session 3 - Problems 13–18

**Hand-in exercises**

Deadline November 19 - Hand-in problems 1–3
Deadline December 25 - Hand-in problems 4–6
Deadline December 4 - Hand-in problems 7–9

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