Modern Cosmology 2005

Exercise set 1

To be solved in exercise session 1

1. *Cosmological distances I.*
A galaxy is observed at a redshift of $z = 0.25$. How distant is this object according to Hubble's law? How accurate is Hubble's law for estimating the luminosity distance at this redshift, under the assumption of a cosmological model with $\Omega_{\rm M} = 0.3$ and $\Omega_\Lambda = 0.7$?

2. *Cosmological distances II.*
Consider a light source at a redshift of $z = 3$ in an Einstein-de Sitter Universe.
a) How far has the light from this object traveled to reach us?
b) How distant is this object today?
c) How distant was this object at the epoch when the light was emitted?
d) How far would light emitted from us today have to travel to reach the object?

3. *Age of the Universe.*
Starting from the Friedmann equations, derive an expression for the age of the Universe $t(z)$, in the case of a standard cosmological model including the parameters $\Omega_{\rm rad}$, $\Omega_{\rm M}$, Ω_Λ, $\Omega_{\rm tot}$.
a) What does the relation for $t(z)$ reduce to in the case of an empty (Milne) Universe?
b) What does the relation for $t(z)$ reduce to in the case of an Einstein-de Sitter Universe?
c) What is the current age of the Universe if $\Omega_{\rm rad} = 8.4 \times 10^{-5}$, $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$, $\Omega_{\rm tot} \approx 1.0$? What was it at redshift $z = 1$?

4. *Matter-radiation equality.*
Estimate the redshift, age and photon temperature of the Universe at the epoch of matter-radiation equality, assuming $\Omega_{\rm rad} = 8.4 \times 10^{-5}$ and $\Omega_{\rm M} = 0.3$.

5. *Inflation I.*
One of the cosmological problems solved by inflation is the flatness problem, i.e. why $|\Omega_{\rm tot} - 1|$ is so small today. For instance, in the currently favoured cosmological model, $|\Omega_{\rm tot} - 1|$ would have to be $|\Omega_{\rm tot} - 1| < 10^{-60}$ at the Planck time to ensure $|\Omega_{\rm tot} - 1| < 0.1$ today, in the absence of inflation, which seems like a case of extreme fine-tuning.

Derive the required $|\Omega_{\rm tot} - 1|$ at $z = 10$ in order to meet the $|\Omega_{\rm tot} - 1| < 0.1$ criterion today, in the case of a Universe dominated by a cosmic fluid with
a) an equation of state $p = 0$ (non-relativistic matter),
b) an equation of state $p = (1/3)\rho$ (radiation),
c) an equation of state $p = -\rho$ (cosmological constant).

6. *Inflation II.*
Consider an inflaton field ϕ with associated potential energy $V(\phi)$. In the slow-roll approximation, the Friedmann equations reduce to:

$$H^2 \approx \frac{8\pi}{3m_{\rm pl}^2} V(\phi), \tag{1}$$

$$3H\dot{\phi} \approx -V', \tag{2}$$

where $V' = {\rm d}V/{\rm d}\phi$. One often defines a slow-roll parameter $\epsilon(\phi)$, with

$$\epsilon(\phi) = \frac{m_{\rm pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2. \tag{3}$$

If we define inflation as an epoch of cosmic acceleration ($\ddot{a} > 0$), then what is the condition on $\epsilon(\phi)$ and ϕ for inflation to occur, if the inflaton potential is assumed to be that of a scalar field

$$V(\phi) = \frac{m^2\phi^2}{2}? \tag{4}$$

Hand-in exercises (deadline September 30)

7. *Optical depth on cosmic scales.*
Assuming the Universe to be filled with a non-evolving population of spherical objects (e.g. galaxies) with $\Omega_{\mathrm{objects}} = 0.1$, individual masses $M_{\mathrm{object}} = 10^{12}\ M_\odot$ and radii $R_{\mathrm{object}} = 10$ kpc, what is the probability of having such an object located along the line of sight to some point-like light source at redshift $z = 1$? You may assume that we live in an $\Omega_{\mathrm{M}} = 0.3$, $\Omega_\Lambda = 0.7$, $H_0 = 72$ km s^{-1} Mpc^{-1} Universe.

8. *The time-dependence of redshift.*
Suppose we were to measure the wavelength of the Lyman alpha line (rest wavelength 1216 Å) in some $z = 6.0$ light source today, and then return to it 100 years later to redo the measurement. How much would we expect the wavelength of the line to have shifted due to cosmic expansion during this time? Is this an observable effect? You may assume that we live in an $\Omega_{\mathrm{M}} = 0.3$, $\Omega_\Lambda = 0.7$, $H_0 = 72$ km s^{-1} Mpc^{-1} Universe.

9. *Inflation III.* The amount of inflation is often specified by the number of e-foldings, N:

$$N = \ln\frac{a(t_{\mathrm{stop}})}{a(t_{\mathrm{start}})} = \int_{t_{\mathrm{start}}}^{t_{\mathrm{stop}}} H \mathrm{d}t, \tag{5}$$

where the integration is carried out from the start t_{start} to the end t_{stop} of the inflationary epoch. Use the slow-roll approximation in exercise 6 to derive an expression for N as a function of ϕ_{start} and ϕ_{stop} in the case of a scalar field potential of the form given by (4).

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