Introduction to Numerical Hydrodynamics and Radiative Transfer

Part II: Hydrodynamics, Lecture 2

HT 2012

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Introduction to Numerical Hydrodynamics

2. The Linear Advection Equation

2.1 Introduction of the Linear Advection Equation

Equations of Fluid Dynamics: Euler Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j + P \delta_{ij}) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \nabla \cdot \left((\frac{1}{2}\rho u^2 + \rho e + P)\boldsymbol{u} \right) = 0$$

describe the conservation of mass, momentum and energy, based on simplifying physical assumptions:

- no external forces (e.g. gravity, radiation pressure)
- no heating or cooling by radiation or heat conduction
- no viscosity (friction at microscopic level, shear)

Equations of Fluid Dynamics: Euler Equations

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mathematically speaking:

- a coupled system of (up to) 5 non-linear PDEs
- dependent on time and (up to) 3 space dimensions
- hyperbolic conservation laws

Equations of Fluid Dynamics: Euler Equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho u\right) + \frac{\partial}{\partial x}\left(\rho u u + P\right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \frac{\partial}{\partial x}\left(\left(\frac{1}{2}\rho u^2 + \rho e + P\right)u\right) = 0$$

... in one space dimension, to simplify the following discussion

- a coupled system of 3 non-linear PDEs
- dependent on time and 1 space dimension
- hyperbolic conservation laws

The Euler Equations in Conservation Form

$$\frac{\partial}{\partial t}\boldsymbol{q} + \frac{\partial}{\partial x}\boldsymbol{F}(\boldsymbol{q}) = 0$$

$$\boldsymbol{q} = \begin{pmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho u^2 + \rho e \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} \rho u \\ \rho u u + P \\ (\frac{1}{2}\rho u^2 + \rho e + P) u \end{pmatrix}$$

... in one space dimension, to simplify the following discussion

- a coupled system of 3 non-linear PDEs
- dependent on time and 1 space dimension
- hyperbolic conservation laws

Mathematical Properties of PDEs

$$\frac{\partial}{\partial t}\,\boldsymbol{q} + \frac{\partial}{\partial x}\,\boldsymbol{F}(\boldsymbol{q}) = 0$$

is a hyperbolic system if the Jacobian matrix of the flux function

$$\boldsymbol{F}'(\boldsymbol{q}) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{q}}$$

has the following property: For each value of $\boldsymbol{q}$ the eigenvalues of $\boldsymbol{F}'(\boldsymbol{q})$ are real, and the matrix is diagonalizable, i.e., there is a complete set of linearly independent eigenvectors.

$\Rightarrow$ linearising about a state $\boldsymbol{q}_0$

$$\frac{\partial}{\partial t}\,(\boldsymbol{q}-\boldsymbol{q}_0) + \boldsymbol{F}'(\boldsymbol{q}_0)\,\frac{\partial}{\partial x}\,(\boldsymbol{q}-\boldsymbol{q}_0) = 0$$

Mathematical Properties of PDEs

classification for a linear second-order differential equation in two independent variables

$$a\frac{\partial^2 q}{\partial x^2} + b\frac{\partial^2 q}{\partial x \partial y} + c\frac{\partial^2 q}{\partial y^2} + d\frac{\partial q}{\partial x} + e\frac{\partial q}{\partial y} + f q = g$$

depends on the sign of the discriminant

$$b^2 - 4ac \begin{cases} < 0 & \text{elliptic} \\ = 0 & \text{parabolic} \\ > 0 & \text{hyperbolic} \end{cases}$$

Mathematical Properties of PDEs

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$$b^2 - 4ac \left[\begin{array}{ll} < 0 & \text{elliptic} \\ = 0 & \text{parabolic} \\ > 0 & \text{hyperbolic} \end{array} \right.$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} = \rho(x, y)$$

Poisson equation

Mathematical Properties of PDEs

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$$b^2 - 4ac \left[\begin{array}{ll} < 0 & \text{elliptic} \\ = 0 & \text{parabolic} \\ > 0 & \text{hyperbolic} \end{array} \right.$$

$$\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2}$$

diffusion equation

Mathematical Properties of PDEs

classification for a linear second-order differential equation in two independent variables

$$a\frac{\partial^2 q}{\partial x^2} + b\frac{\partial^2 q}{\partial x \partial y} + c\frac{\partial^2 q}{\partial y^2} + d\frac{\partial q}{\partial x} + e\frac{\partial q}{\partial y} + f q = g$$

depends on the sign of the discriminant

$$b^2 - 4ac \left[\begin{array}{ll} < 0 & \text{elliptic} \\ = 0 & \text{parabolic} \\ > 0 & \text{hyperbolic} \end{array} \right.$$

$$\frac{\partial^2 \rho_1}{\partial t^2} - a_0^2 \frac{\partial^2 \rho_1}{\partial x^2} = 0$$

wave equation

Waves: Small-Amplitude Sound Waves

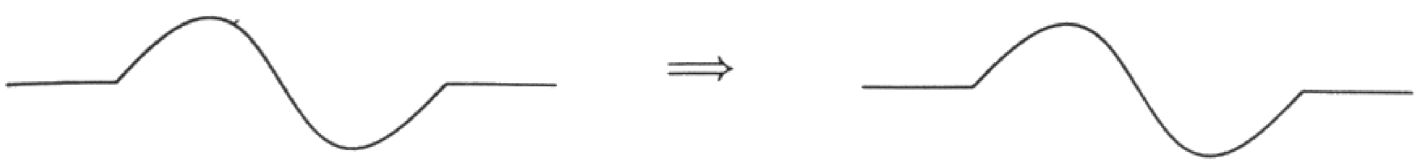
linearisation of Euler equations
$\Rightarrow$ homogeneous wave equation:

$$\frac{\partial^2 \rho_1}{\partial t^2} - a_0^2 \frac{\partial^2 \rho_1}{\partial x^2} = 0$$

general solution:

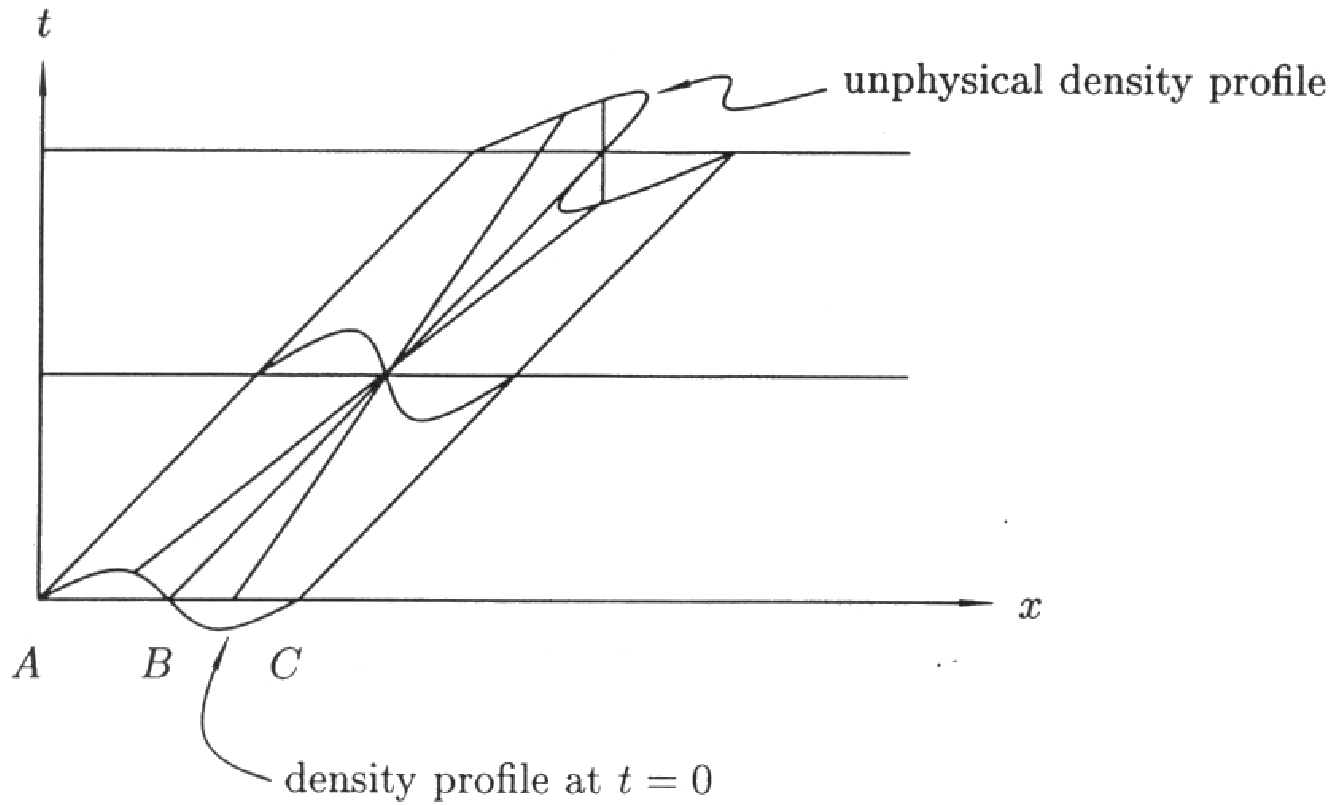
$$\rho_1 = f(x - a_0 t) + g(x + a_0 t)$$

waves propagating with sound speed a_0



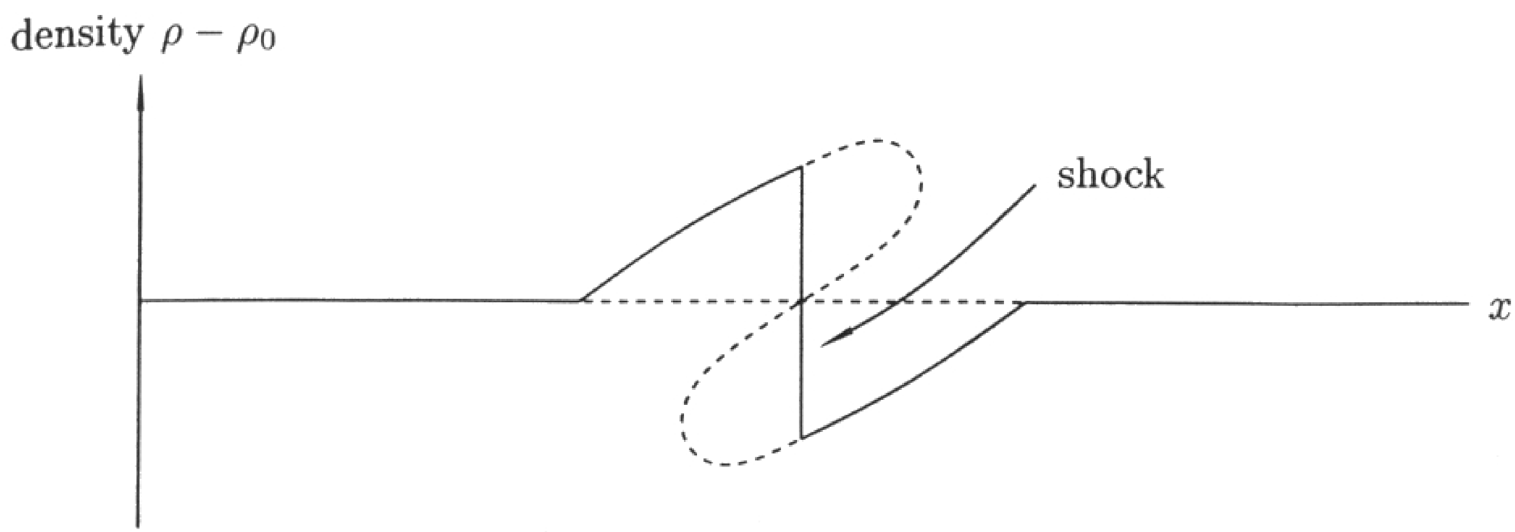
In the absence of dissipation and spatial inhomogeneities (or dispersion), the waveform of a disturbance governed by a linear wave equation maintains its size and shape forever, apart from propagation at a constant wave speed.

Waves: Steepening of Acoustic Waves



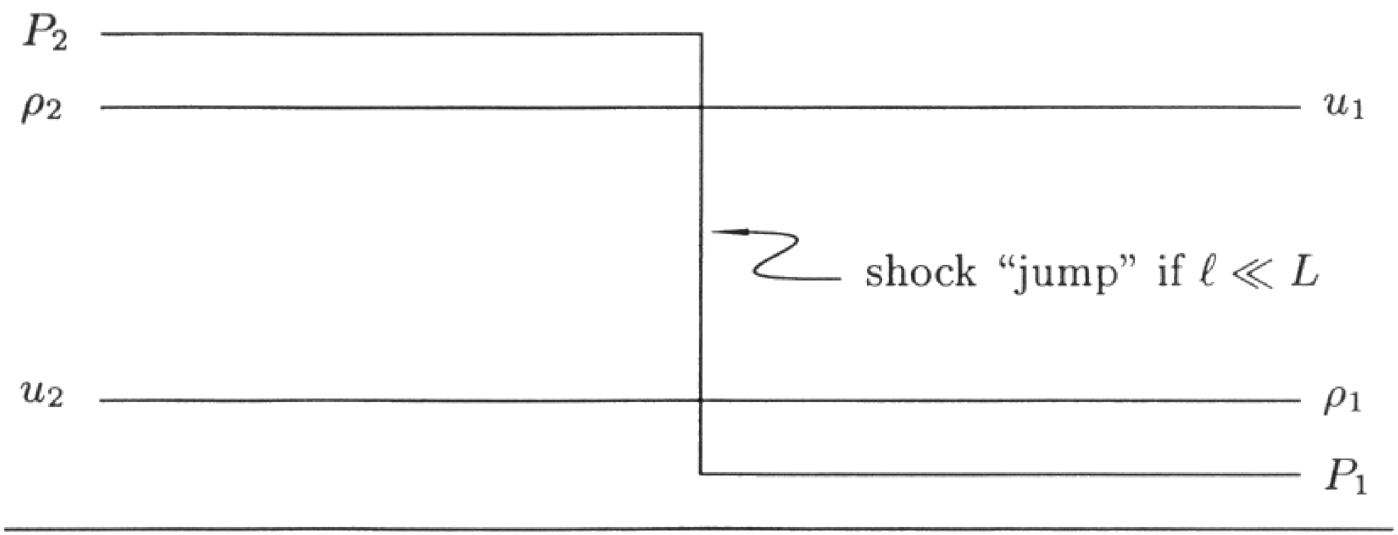
An acoustic wave of finite amplitude, even if it starts with a perfect sinusoidal shape and propagates in an undisturbed medium of exactly uniform properties, would inevitably steepen in its waveform.

Waves: Steepening of Acoustic Waves



The tendency for nonlinearities to steepen the wave profile, which would produce multiple values for fluid properties such as gas density and velocity, must be eventually offset by the onset of strong viscous forces. The balance of the viscous forces and the steepening tendency mediates a shock, which is approximated in ideal fluid flow as a discontinuous jump of gas properties across the front.

Waves: Structure of Shock Waves



On macroscopic scales, shock transitions may be approximated as single discontinuous jumps.

jump conditions $\Leftrightarrow$ conservation laws

Properties of the Euler Equations

Some fundamental properties of the Euler equations may create difficulties for numerical solvers and make the analysis of discretization schemes more complicated:

- depending on the number of spatial dimensions, we have 3-5 coupled PDEs which need to be solved simultaneously
- non-linearity: even with smooth initial conditions the solutions have a tendency to develop discontinuities (shocks)

Goal:

Find simple limit cases (fewer equations, linear, if possible) which allows us to test certain aspects of numerical schemes before attacking the Euler equations in their full glory.

Euler Equations, Special Case: Passive Advection

The Euler equations, 1D:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

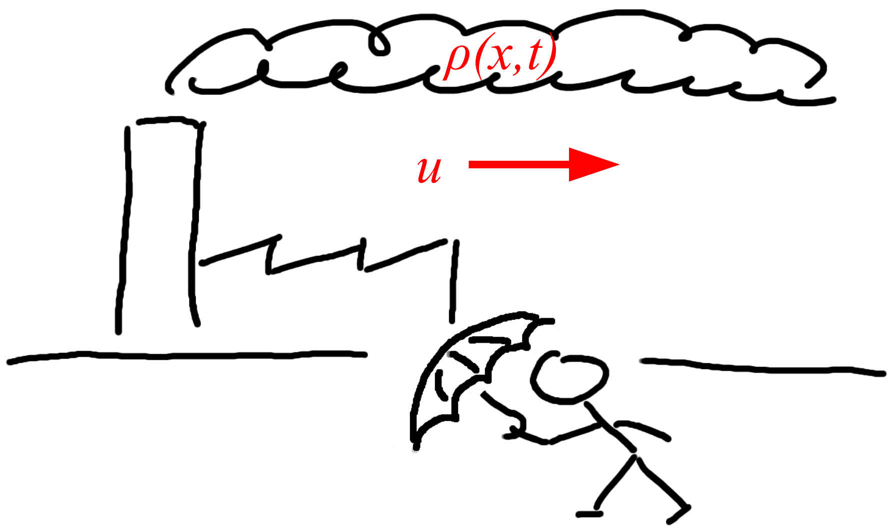
$$\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u + P) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \frac{\partial}{\partial x}\left((\frac{1}{2}\rho u^2 + \rho e + P)u\right) = 0$$

Assuming constant pressure P and a constant flow velocity u, the system of the Euler equations reduces to 2 separate linear PDEs:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0 \qquad\qquad \frac{\partial e}{\partial t} + u\frac{\partial e}{\partial x} = 0$$

Euler Equations, Special Case: Passive Advection



An equation of this type is known as linear advection equation and describes the passive transport of the quantity ρ in a flow with given constant velocity u (no influence of ρ on the flow):

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0$$

Euler Equations, Special Case: Passive Advection

The linear advection equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0$$

with constant flow velocity u
and initial condition (at $t = 0$)

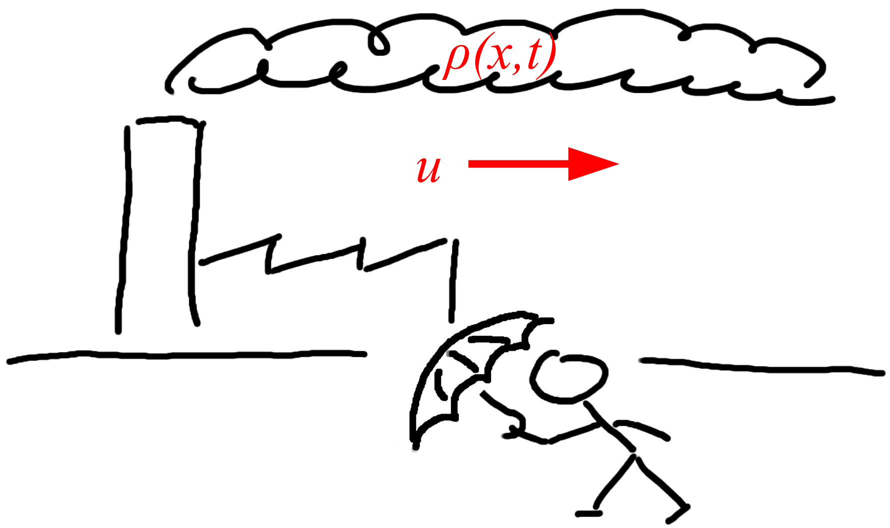
$$\rho(x, 0) = \rho_0(x)$$

has the general solution

$$\rho(x, t) = \rho_0(x - ut)$$

which is constant along the characteristics $c(t) = x_0 + ut$

Proof by checking ...

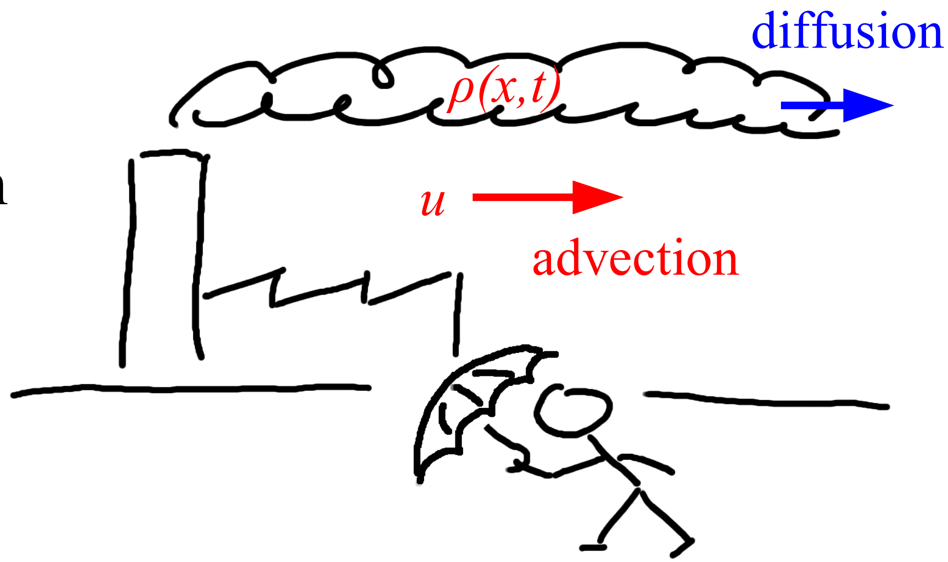


Modification: Advection-Diffusion Equation

A relative of the linear advection equation (useful for numerics*):

The advection-diffusion equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(u\rho - D\frac{\partial \rho}{\partial x}\right) = 0$$



advective flux (transport due to fluid motion)

diffusive flux (diffusion due to gradient in ρ)

with diffusion coefficient $D > 0$

* parabolic PDE, smooth solution!

for constant u and D:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = D\frac{\partial^2 \rho}{\partial x^2}$$

Introduction to Numerical Hydrodynamics

2. The Linear Advection Equation

2.2 Discretization Attempts and Basic Concepts

The Diffusion Equation

The diffusion equation (describing, e.g., heat conduction)

$$\frac{\partial y}{\partial t} = D \frac{\partial^2 y}{\partial x^2}$$

is a parabolic PDE. It always has a smooth solution for $t > 0$ even if the initial conditions are discontinuous.

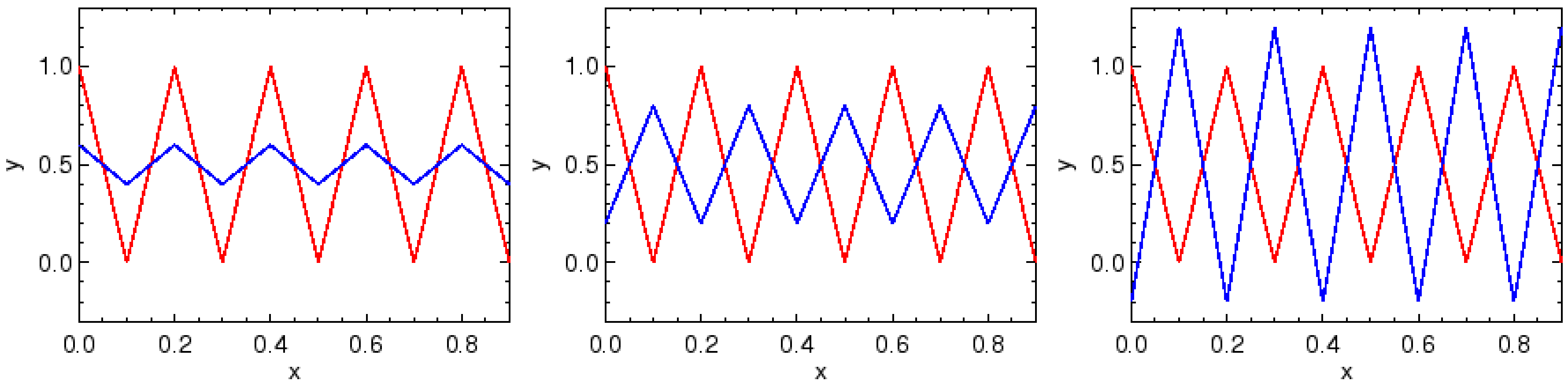
Discretization in time and space: $t^n = n\Delta t + t_0 \quad x_i = i\Delta x + x_0$

$$\frac{\partial y}{\partial t} \to \frac{y_i^{n+1} - y_i^n}{\Delta t} \qquad \frac{\partial^2 y}{\partial x^2} \to \frac{y_{i+1}^n - 2y_i^n + y_{i-1}^n}{\Delta x^2}$$

$$\Rightarrow y_i^{n+1} = y_i^n + \frac{\Delta t}{\Delta x^2} D \left(y_{i+1}^n - 2y_i^n + y_{i-1}^n \right)$$

explicit Euler scheme

The Diffusion Equation



Figures courtesy of Bernd Freytag

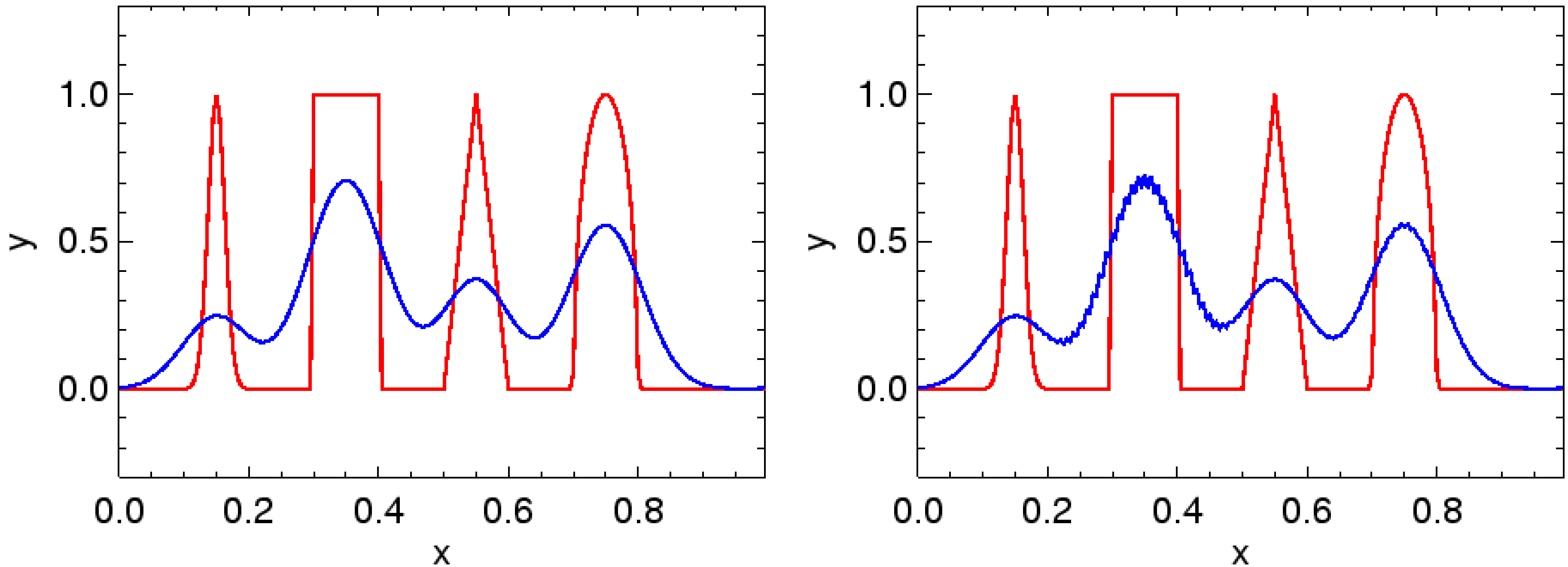
Stability of Euler scheme for the diffusion equation:
Initial condition (red) and solution (blue) for $D\Delta t/\Delta x^2 = 0.2$ (left), 0.4 (middle) and 0.6 (right)

stability: $D\Delta t/\Delta x^2 = 1/2$ positivity: $D\Delta t/\Delta x^2 = 1/4$

$$y_i^{n+1} = y_i^n + \frac{\Delta t}{\Delta x^2} D \left(y_{i+1}^n - 2y_i^n + y_{i-1}^n \right)$$

explicit Euler scheme

The Diffusion Equation



Figures courtesy of Bernd Freytag

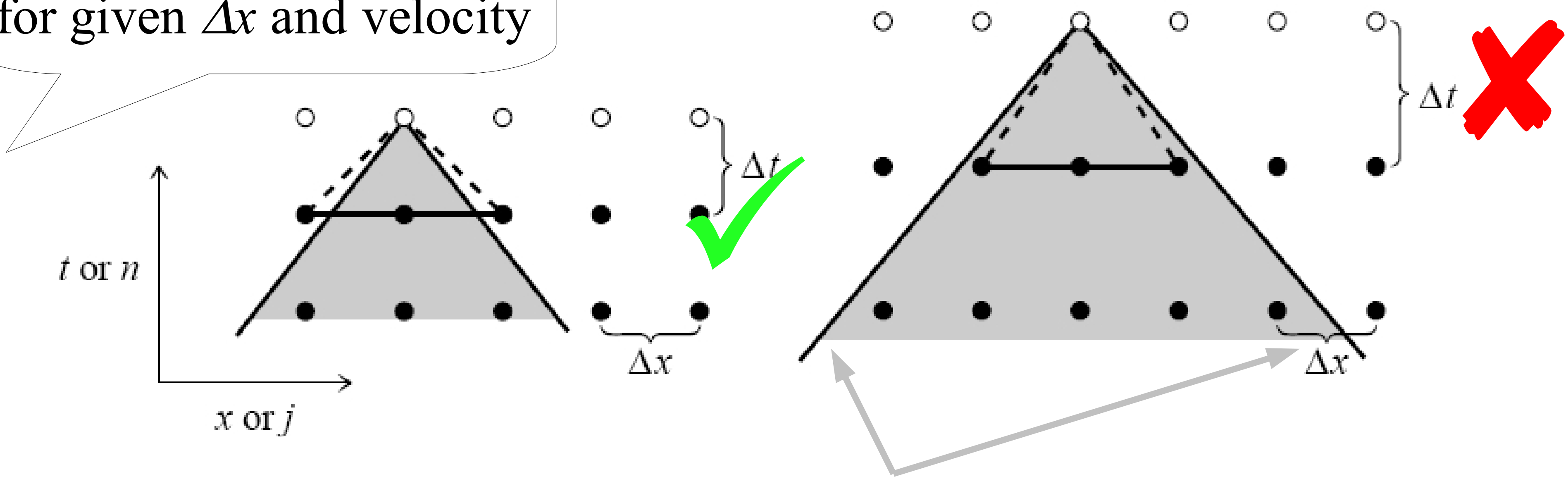
Initial condition (red) and solution (blue) for $D\Delta t/\Delta x^2 = 0.1$ (left) and *0.5* (right) after 500 and 100 time steps, respectively.

The slightly too large time step causes non-decaying spurious oscillations in the right panel.

The simple discretization gives reasonable results.

Domain of Dependence and CFL Condition

CFL condition limits Δt for given Δx and velocity

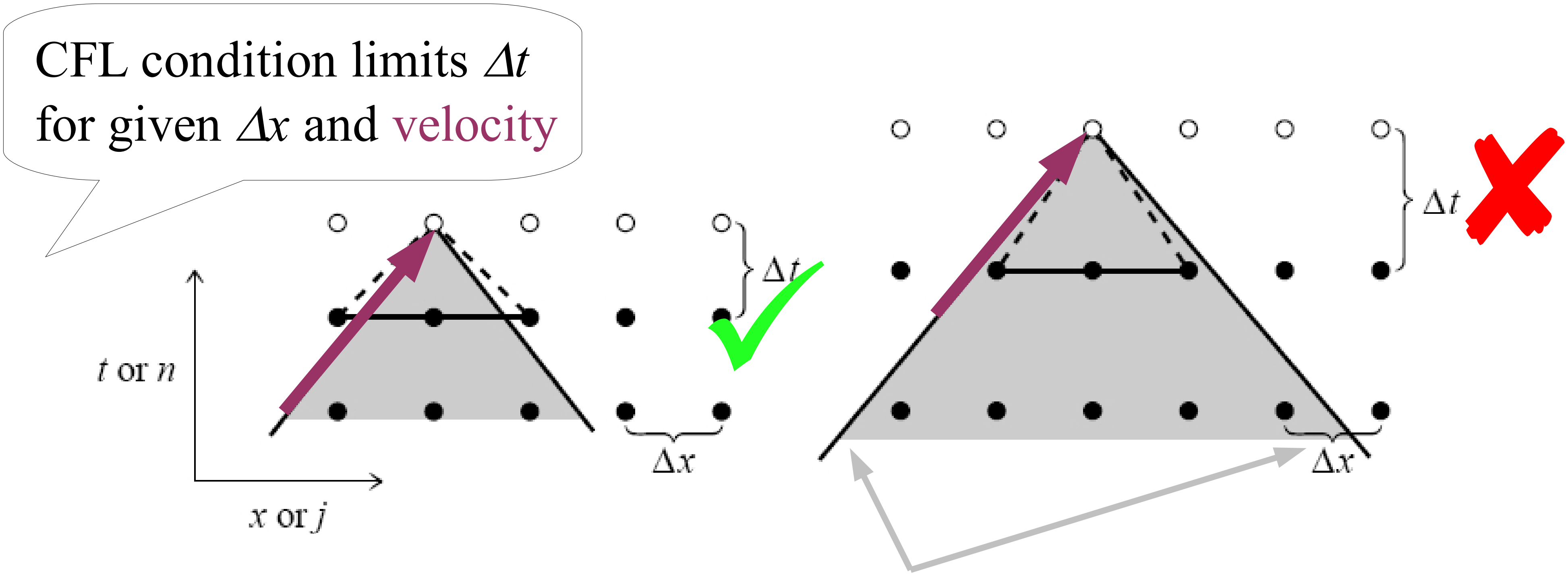


Hyperbolic PDEs have a finite physical domain of dependence due to the finite travelling speed of waves (↔ characteristics).

Courant-Friedrichs-Levy condition (CFL condition):
The numerical domain of dependence must contain the physical domain of dependence.

The CFL condition is necessary for stability, but not sufficient.

Domain of Dependence and CFL Condition



Hyperbolic PDEs have a finite physical domain of dependence due to the finite travelling speed of waves (↔ characteristics).

Example: linear advection equation $|u\Delta t / \Delta x| \leq 1$

constant flow velocity (everywhere, at all times) →
characteristics = straight lines, slope corresponding to velocity
physical domain of dependence = starting point of characteristic

The Linear Advection Equation

The linear advection equation (with constant flow velocity u)

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0$$

has a known analytical solution (pure transport of initial profile)
$\Rightarrow$ perfect for testing numerical schemes!

Discretization in time and space: $t^n = n\Delta t + t_0$ $\quad x_i = i\Delta x + x_0$

$$\frac{\partial \rho}{\partial t} \rightarrow \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} \qquad \frac{\partial \rho}{\partial x} \rightarrow \frac{\rho_{i+1}^n - \rho_{i-1}^n}{2\Delta x}$$

$$\Rightarrow \rho_i^{n+1} = \rho_i^n - u \frac{\Delta t}{2\Delta x} \left(\rho_{i+1}^n - \rho_{i-1}^n \right)$$

explicit Euler scheme (FTCS)

The Linear Advection Equation – Crash

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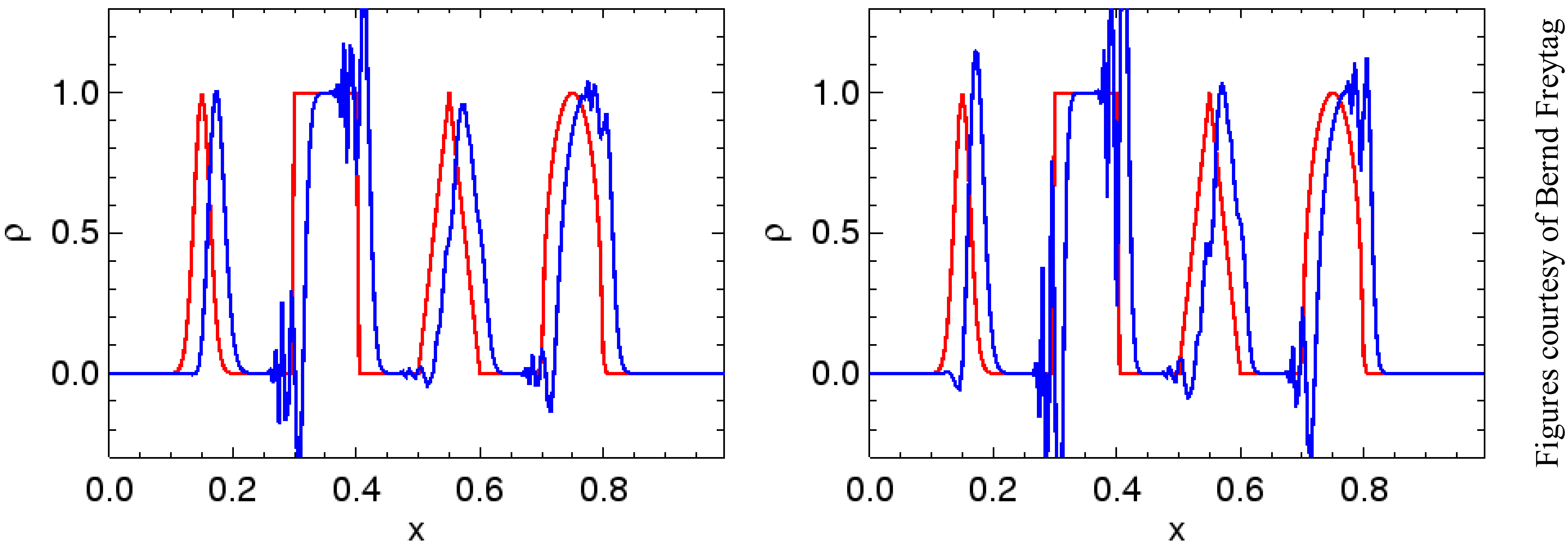
explicit Euler scheme (FTCS)

UNSTABLE !

... even if CFL condition ($|u\Delta t / \Delta x| \le 1$) is fulfilled ...

The Linear Advection Equation – Crash



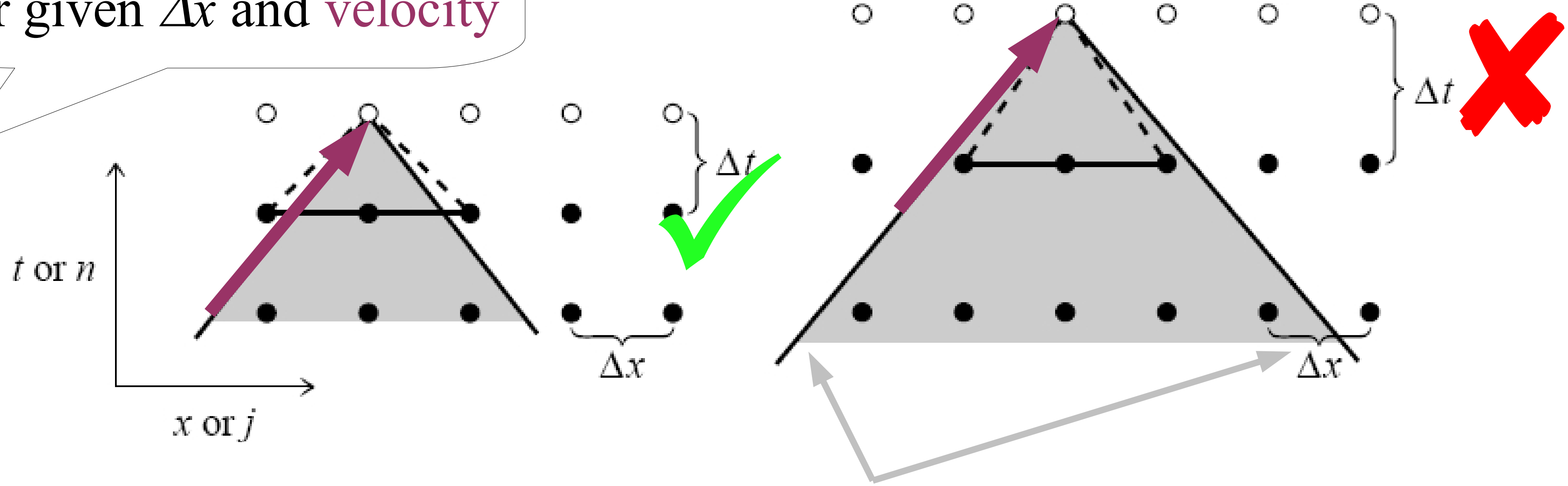
Initial condition (red) and solution (blue) for $u\Delta t/\Delta x = 0.1$ (left) and 0.5 (right) after 50 and 10 time steps, respectively.

The growing oscillations render the scheme useless.

Linear stability analysis shows: The explicit Euler scheme (FTCS) is unconditionally unstable.

Domain of Dependence and CFL Condition

CFL condition limits Δt for given Δx and velocity



Hyperbolic PDEs have a finite physical domain of dependence due to the finite travelling speed of waves (↔ characteristics).

Example: linear advection equation $\qquad |u\Delta t / \Delta x| \leq 1$

constant flow velocity (everywhere, at all times) →
characteristics = straight lines, slope corresponding to velocity
physical domain of dependence = starting point of characteristic

The Linear Advection Equation – A New Hope

The linear advection equation (with constant flow velocity u)

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Discretization in time and space: $t^n = n\Delta t + t_0 \quad x_i = i\Delta x + x_0$

$$\frac{\partial \rho}{\partial t} \rightarrow \frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} \qquad \frac{\partial \rho}{\partial x} \rightarrow \frac{\rho_i^n - \rho_{i-1}^n}{\Delta x} \quad (u > 0)$$

$$\Rightarrow \rho_i^{n+1} = \rho_i^n - u \frac{\Delta t}{\Delta x} \left(\rho_i^n - \rho_{i-1}^n \right)$$

donor cell
(upwind)
scheme (FTBS)

The Linear Advection Equation – A New Hope

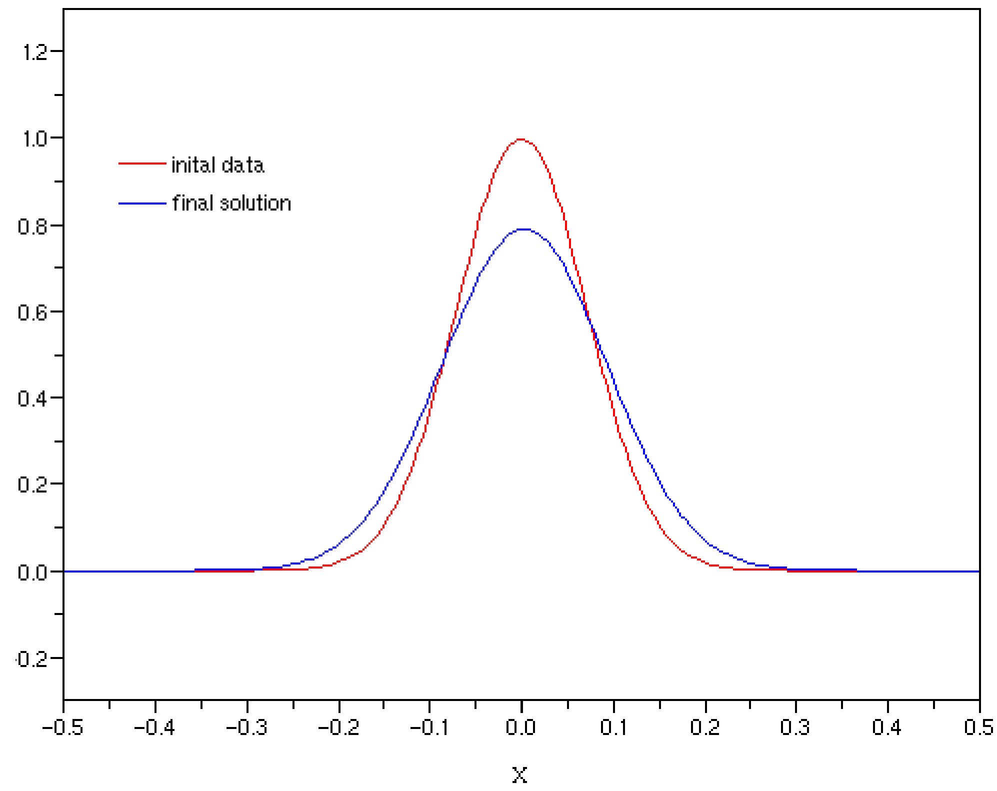
numerical solution with

$$u\Delta t/\,\Delta x = 0.4$$

(Courant Number) and periodic boundary conditions, plotted after one cycle (blue curve), for Gaussian initial profile

numerical parameters:

200 grid points
500 time steps



$$\rho_i^{n+1} = \rho_i^n - u\frac{\Delta t}{\Delta x}\left(\rho_i^n - \rho_{i-1}^n\right)$$

donor cell (upwind) scheme (FTBS)

The Linear Advection Equation – A New Hope

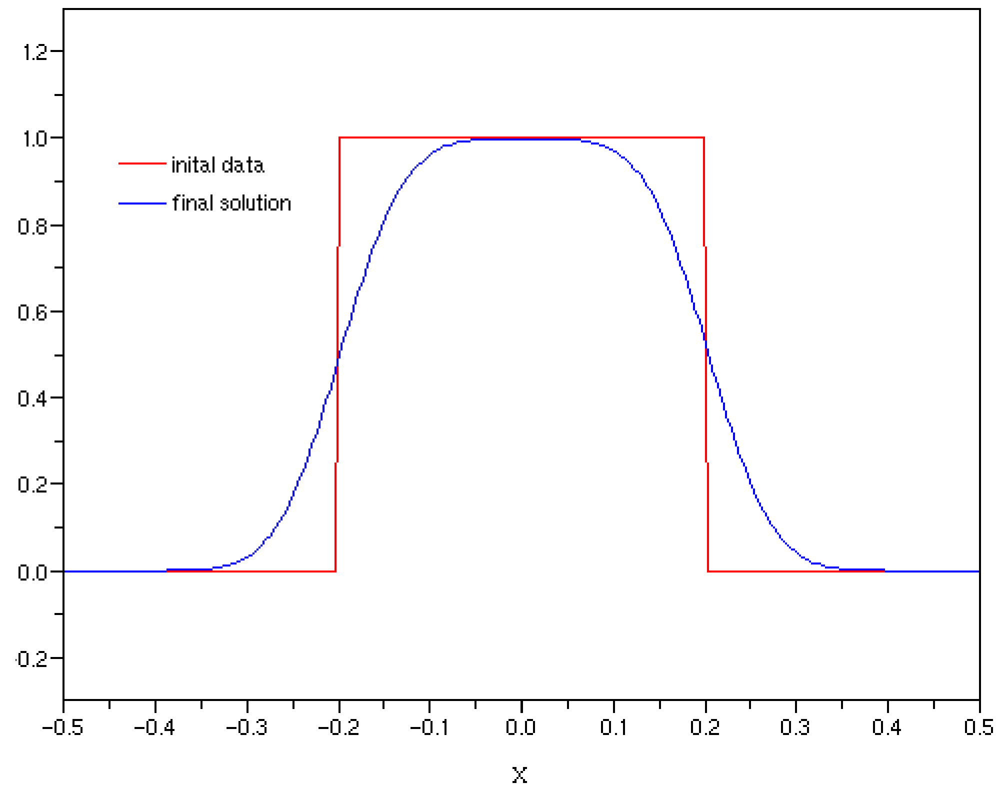
numerical solution with

$u\Delta t/ \Delta x = 0.4$

(Courant Number) and periodic boundary conditions, plotted after one cycle (blue curve), for box initial profile

numerical parameters:

200 grid points
500 time steps



$$\rho_i^{n+1} = \rho_i^n - u\frac{\Delta t}{\Delta x}\left(\rho_i^n - \rho_{i-1}^n\right)$$

donor cell (upwind) scheme (FTBS)

The Linear Advection Equation – Homework 1

Compute the numerical solution of the linear advection equation

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0$$

using the upwind (donor cell) scheme*, with the initial conditions

$$\rho_0(x) = 1 \quad \text{for} \quad x \leq 0 \quad \text{and} \quad \rho_0(x) = 0 \quad \text{for} \quad x > 0$$

in the range $0 \leq x \leq 1$, assuming $u=1$ and $\Delta t/\Delta x = 0.5$

Plot the resulting numerical solution and the exact analytical solution at time $t = 0.5$ for the cases $\Delta x = 0.01$ and $\Delta x = 0.0025$.

*upwind (donor cell) scheme $\Rightarrow \rho_i^{n+1} = \rho_i^n - u\dfrac{\Delta t}{\Delta x}\left(\rho_i^n - \rho_{i-1}^n\right)$