Introduction to Numerical Hydrodynamics and Radiative Transfer

Part II: Hydrodynamics, Lecture 4

HT 2012

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Introduction to Numerical Hydrodynamics

2. The Linear Advection Equation

2.5 Analysis of Numerical Schemes

The Linear Advection Equation – Homework 2

Compute the numerical solution of the linear advection equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0$$

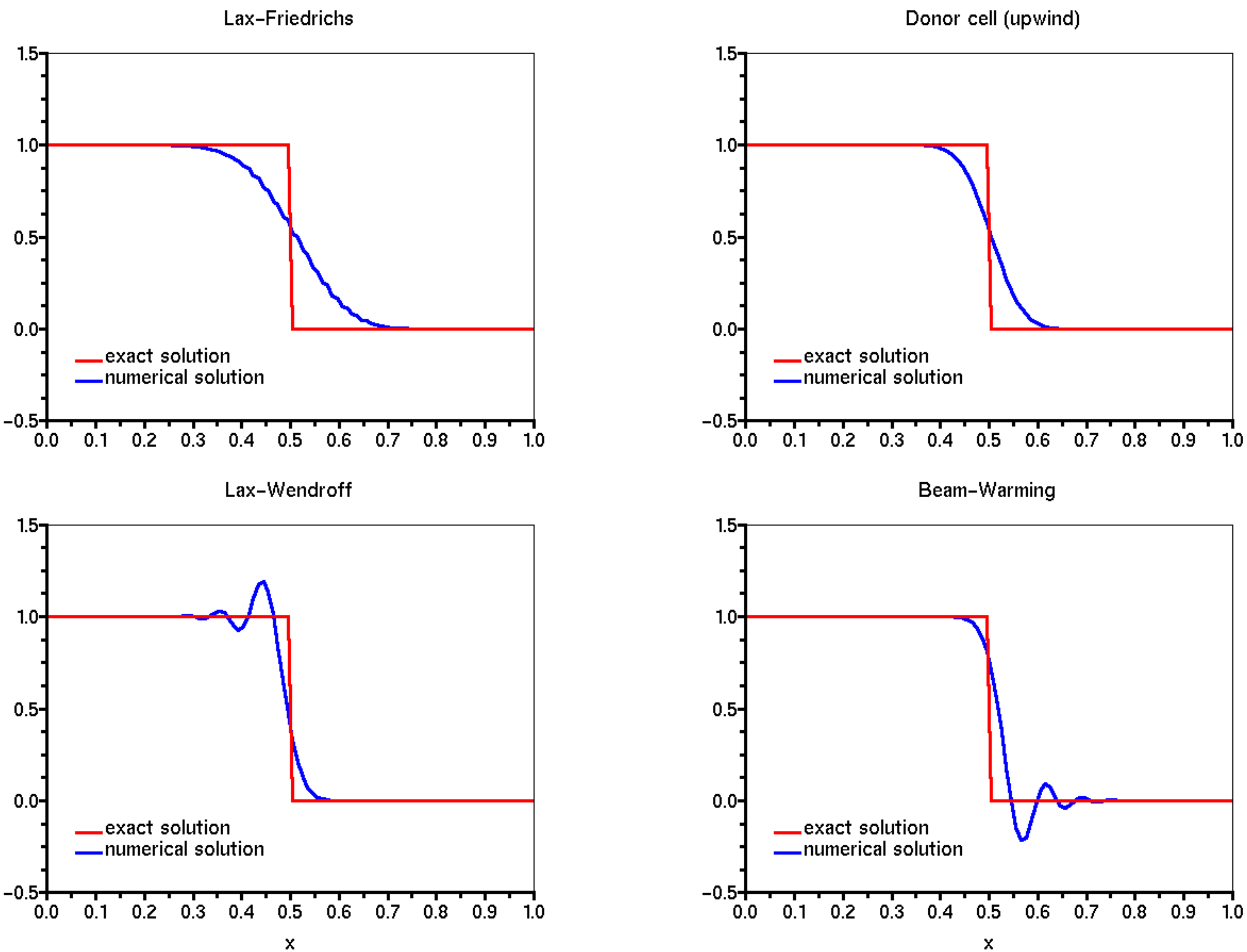
in the range $0 \leq x \leq 1$, with the initial conditions

$$\rho_0(x) = 1 \quad \text{for} \quad x<0 \qquad \text{and} \qquad \rho_0(x) = 0 \quad \text{for} \quad x>0$$

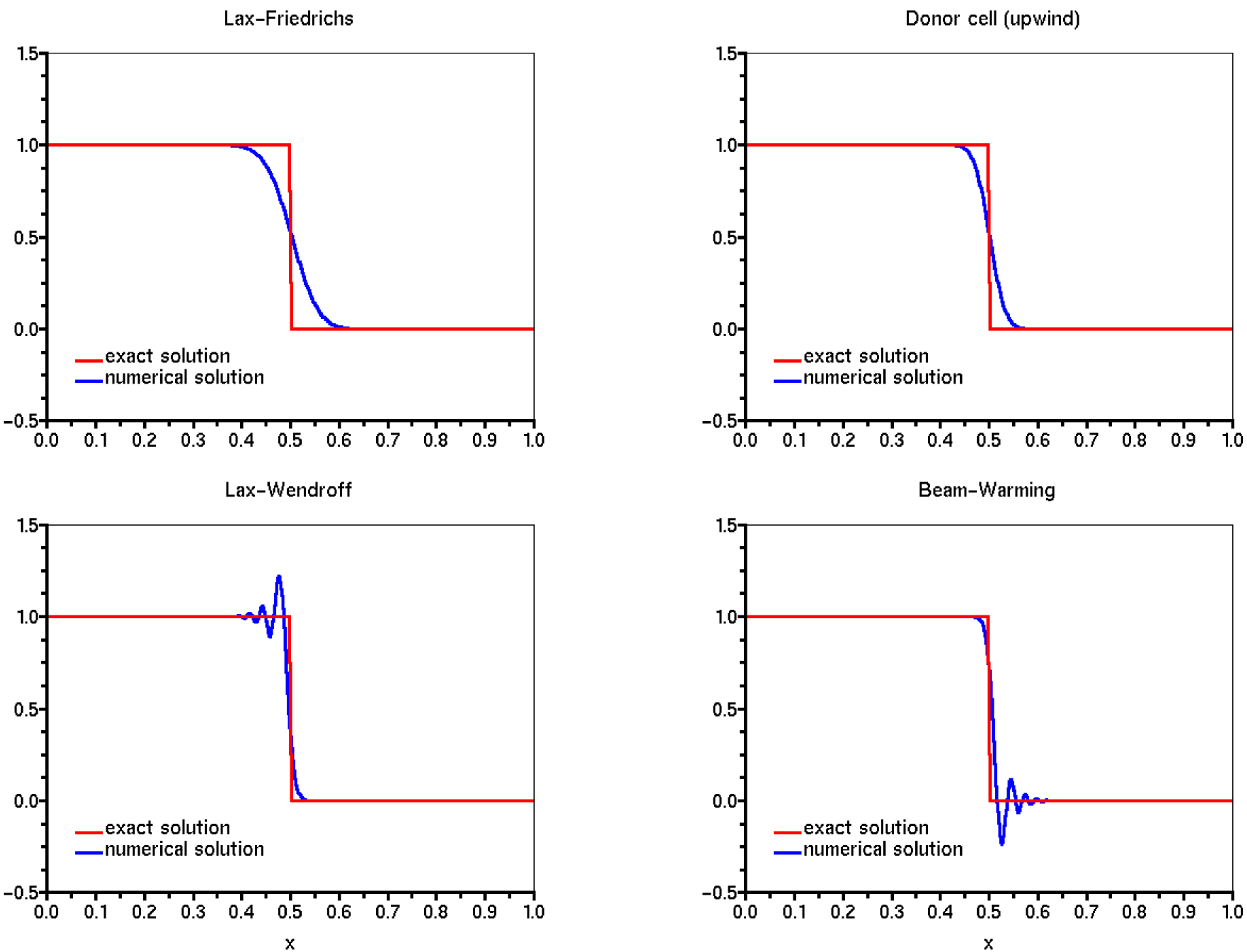
assuming $u=1$ and $\Delta t/\Delta x = 0.5$, for different numerical schemes: (a) Lax-Friedrichs, (b) Lax-Wendroff and (c) Beam-Warming.

Plot the resulting numerical solutions and the exact analytical solution at time $t = 0.5$ for the cases $\Delta x = 0.01$ and $\Delta x = 0.0025$, and compare the results to the upwind (donor cell) scheme (see homework 1).

Homework 1 and 2 – Results



Homework 1 and 2 – Results



Homework 1 and 2 – Summary

- Higher spatial and temporal resolution (for given velocity and Courant number) gives a steeper gradient in the numerical solution (i.e., a better correspondence to the exact solution) .
- The first order methods (Lax-Friedrichs and upwind/donor cell) give very smeared solutions.
- The second order methods (Lax Wendroff and Beam-Warming) give oscillations.

This qualitatively different behaviour of first and second order methods is typical, and can be understood with the following analysis using modified equations.

Analysis of Schemes – Modified Equations

General idea:

The discrete equation is an approximation of the original PDE (recall definitions of truncation error and consistency of a scheme)

BUT:

The discrete equation may be an even better approximation of a modified version of the original PDE (corresponding to a higher order of the truncation error).

This modified equation may tell us something about the qualitative behaviour of the numerical scheme ...

Truncation Error

A sufficiently smooth function can be expanded in a Taylor series:

$$\rho(x+\Delta x,t)=\sum_{l=0}^{\infty}\frac{1}{l!}\left.\frac{\partial^l\rho}{\partial x^l}\right|_{x,t}\Delta x^l=\rho(x,t)+\left.\frac{\partial\rho}{\partial x}\right|_{x,t}\Delta x+O\left(\Delta x^2\right)\quad .$$

Solving for $\frac{\partial\rho}{\partial x}$ gives

$$\frac{\partial\rho}{\partial x}=\frac{\rho(x+\Delta x,t)-\rho(x,t)}{\Delta x}+O\left(\Delta x\right)\quad .$$

Repeating this for the time derivative and applying it to an entire PDE (FTFS) gives

$$\underbrace{\frac{\partial\rho}{\partial t}+v\,\frac{\partial\rho}{\partial x}}_{\text{PDE}}=\underbrace{\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t}+v\,\frac{\rho_{i+1}^n-\rho_i^n}{\Delta x}}_{\text{numerical scheme}}+\underbrace{O(\Delta t,\Delta x)}_{\text{truncation error}}\quad .$$

The order of the truncation error is $O\left(\Delta t,\Delta x\right)$ in this case (FTFS).

A high order of the truncation error hints at good accuracy for smooth functions.

Analysis of Schemes – Modified Equations

Example: Lax-Friedrichs method for the linear advection equation

The discrete equation:

$$\frac{1}{\Delta t}\left(\rho_i^{n+1} - \frac{1}{2}\left(\rho_{i-1}^n + \rho_{i+1}^n\right)\right) + \frac{u}{2\Delta x}\left(\rho_{i+1}^n - \rho_{i-1}^n\right) = 0$$

Replacing the discrete solution ρ_i^n with the exact solution $\rho(x,t)$ of the original PDE, we obtain the local truncation error for this numerical scheme:

$$L_{\Delta t,\Delta x}(x,t) = \frac{1}{\Delta t}\left(\rho(x,t+\Delta t) - \frac{1}{2}\left(\rho(x-\Delta x,t)+\rho(x+\Delta x,t)\right)\right)$$

$$+ \frac{u}{2\Delta x}\left(\rho(x+\Delta x,t) - \rho(x-\Delta x,t)\right)$$

Analysis of Schemes – Modified Equations

Example: Lax-Friedrichs method for the linear advection equation

Taylor expansion about $\rho(x,t)$ gives:

$$L_{\Delta t,\Delta x}(x,t) = \frac{1}{\Delta t}\left(\left(\rho + \Delta t \frac{\partial \rho}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 \rho}{\partial t^2} + \ldots\right) - \left(\rho + \frac{1}{2}\Delta x^2 \frac{\partial^2 \rho}{\partial x^2} + \ldots\right) \right)$$

$$+ \frac{u}{2\Delta x}\left(2\Delta x \frac{\partial \rho}{\partial x} + \frac{1}{3}\frac{\partial^3 \rho}{\partial x^3} + \ldots \right)$$

$$L_{\Delta t,\Delta x}(x,t) = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \frac{1}{2}\left(\Delta t \frac{\partial^2 \rho}{\partial t^2} - \frac{\Delta x^2}{\Delta t}\frac{\partial^2 \rho}{\partial x^2} \right) + O(\Delta x^2)$$

Analysis of Schemes – Modified Equations

Example: Lax-Friedrichs method for the linear advection equation

Taylor expansion about $\rho(x,t)$ gives:

$$L_{\Delta t,\Delta x}(x,t) = \frac{1}{\Delta t}\left(\left(\rho + \Delta t \frac{\partial \rho}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 \rho}{\partial t^2} + \ldots\right) - \left(\rho + \frac{1}{2}\Delta x^2 \frac{\partial^2 \rho}{\partial x^2} + \ldots\right) \right)$$

$$+ \frac{u}{2\Delta x}\left(2\Delta x \frac{\partial \rho}{\partial x} + \frac{1}{3}\frac{\partial^3 \rho}{\partial x^3} + \ldots \right)$$

$$L_{\Delta t,\Delta x}(x,t) = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \frac{1}{2}\left(\Delta t \frac{\partial^2 \rho}{\partial t^2} - \frac{\Delta x^2}{\Delta t}\frac{\partial^2 \rho}{\partial x^2} \right) + O(\Delta x^2)$$

$= 0$ (ρ is a solution of the original PDE)

Analysis of Schemes – Modified Equations

Example: Lax-Friedrichs method for the

Taylor expansion about $\rho(x,t)$ gives:

$$\frac{\partial^2 \rho}{\partial t^2} = -u \, \frac{\partial}{\partial t}\frac{\partial \rho}{\partial x} = -u \, \frac{\partial}{\partial x}\frac{\partial \rho}{\partial t} = u^2 \, \frac{\partial^2 \rho}{\partial x^2}$$

$$L_{\Delta t,\Delta x}(x,t) = \frac{1}{\Delta t}\Bigg(\Bigg(\rho + \Delta t\frac{\partial \rho}{\partial t} + \frac{1}{2}\Delta t^2 \ldots \Bigg)\Bigg)$$

$$+ \frac{u}{2\Delta x}\Bigg(2\Delta x\frac{\partial \rho}{\partial x} + \frac{1}{\ldots x^3} + \ldots\Bigg)$$

$$L_{\Delta t,\Delta x}(x,t) = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \frac{1}{2}\left(\Delta t\frac{\partial^2 \rho}{\partial t^2} - \frac{\Delta x^2}{\Delta t}\frac{\partial^2 \rho}{\partial x^2}\right) + O(\Delta x^2)$$

$= 0$ (ρ is a solution of the original PDE)

Analysis of Schemes – Modified Equations

Example: Lax-Friedrichs method for the linear advection equation

Taylor expansion about $\rho(x,t)$ gives:

$$L_{\Delta t,\Delta x}(x,t) = \frac{1}{\Delta t}\left(\left(\rho + \Delta t \frac{\partial \rho}{\partial t} + \frac{1}{2}\Delta t^2 \frac{\partial^2 \rho}{\partial t^2} + \ldots\right) - \left(\rho + \frac{1}{2}\Delta x^2 \frac{\partial^2 \rho}{\partial x^2} + \ldots\right)\right)$$

$$+ \frac{u}{2\Delta x}\left(2\Delta x \frac{\partial \rho}{\partial x} + \frac{1}{3}\frac{\partial^3 \rho}{\partial x^3} + \ldots\right)$$

$$L_{\Delta t,\Delta x}(x,t) = \frac{1}{2}\Delta t \left(u^2 - \frac{\Delta x^2}{\Delta t^2}\right)\frac{\partial^2 \rho}{\partial x^2} + O(\Delta t^2)$$

Analysis of Schemes – Modified Equations

Example: Lax-Friedrichs method for the linear advection equation

The discrete equation:

$$\frac{1}{\Delta t}\left(\rho_i^{n+1} - \frac{1}{2}\left(\rho_{i-1}^n + \rho_{i+1}^n\right)\right) + \frac{u}{2\Delta x}\left(\rho_{i+1}^n - \rho_{i-1}^n\right) = 0$$

Replacing the discrete solution ρ_i^n with the exact solution $\rho(x,t)$ of the original PDE, and using Taylor expansion, we obtain the local truncation error:

$$L_{\Delta t, \Delta x}(x,t) = \frac{1}{2}\Delta t \left(u^2 - \frac{\Delta x^2}{\Delta t^2} \right) \frac{\partial^2 \rho}{\partial x^2} + O\left(\Delta t^2\right)$$

Analysis of Schemes – Modified Equations

Example: Lax-Friedrichs method for the linear advection equation

The discrete equation

$$\frac{1}{\Delta t}\left(\rho_i^{n+1} - \frac{1}{2}\left(\rho_{i-1}^{n} + \rho_{i+1}^{n}\right)\right) + \frac{u}{2\Delta x}\left(\rho_{i+1}^{n} - \rho_{i-1}^{n}\right) = 0$$

represents a first order accurate approximation to the original PDE (i.e. the linear advection equation), but a second order accurate approximation to the modified equation

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = \frac{\Delta x^2}{2\,\Delta t}\left(1 - \left(u\frac{\Delta t}{\Delta x}\right)^2\right)\frac{\partial^2 \rho}{\partial x^2}$$

... which is an advection-diffusion equation

→ smearing of numerical solutions has a simple explanation!

Analysis of Schemes – Modified Equations

The advection-diffusion equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = D \frac{\partial^2 \rho}{\partial x^2}$$

with

$$D = \frac{\Delta x^2}{2 \Delta t} \left(1 - \left(u \frac{\Delta t}{\Delta x} \right)^2 \right) \qquad \text{(Lax-Friedrichs scheme)}$$

or

$$D = \frac{u \Delta x}{2} \left(1 - u \frac{\Delta t}{\Delta x} \right) \qquad \text{(upwind (donor cell) scheme)}$$

is a second order accurate model for the respective discrete versions of the linear advection equation

→ explains typical diffusive behaviour of first order schemes

Analysis of Schemes – Modified Equations

The advection-diffusion equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = D \frac{\partial^2 \rho}{\partial x^2}$$

with

$$D = -\frac{u \Delta x}{2} \left(1 + u \frac{\Delta t}{\Delta x} \right)$$ (FTFS scheme)

or

$$D = \frac{u \Delta x}{2} \left(1 - u \frac{\Delta t}{\Delta x} \right)$$ (upwind (donor cell) scheme)

is a second order accurate model for the respective discrete versions of the linear advection equation

→ *D<0* (anti-diffusion) explains problems with FTFS scheme

Analysis of Schemes – Modified Equations

The dispersive equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = \mu \frac{\partial^3 \rho}{\partial x^3}$$

with

$$\mu = \frac{u \Delta x^2}{6} \left(\left(u \frac{\Delta t}{\Delta x} \right)^2 - 1 \right) \qquad \text{(Lax-Wendroff)}$$

or

$$\mu = \frac{u \Delta x^2}{6} \left(2 - u \frac{3 \Delta t}{\Delta x} + \left(u \frac{\Delta t}{\Delta x} \right)^2 \right) \qquad \text{(Beam-Warming)}$$

is a third order accurate model for the respective discrete versions of the linear advection equation

→ explains typical oscillations of second order schemes ...

Analysis of Schemes – Modified Equations

The dispersive equation

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = \mu \frac{\partial^3 \rho}{\partial x^3}$$

with

$$\mu = \frac{u\Delta x^2}{6}\left(\left(u\frac{\Delta t}{\Delta x}\right)^2 - 1 \right) \qquad \text{(Lax-Wendroff)}$$

or

$$\mu = \frac{u\Delta x^2}{6}\left(2 - u\frac{3\Delta t}{\Delta x} + \left(u\frac{\Delta t}{\Delta x}\right)^2 \right) \qquad \text{(Beam-Warming)}$$

applied to a linear wave with frequency ω and wave number k leads to the dispersion relation, phase velocity and group velocity

$$\omega = u\,k + \mu\,k^3\,, \quad c_p = u + \mu\,k^2 \quad \textit{and} \quad c_g = u + 3\mu\,k^2$$

Analysis of Schemes – Modified Equations

The dispersive equation

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = \mu \, \frac{\partial^3 \rho}{\partial x^3}$$

with

$$\mu < 0 \quad \text{for} \quad \left| u\frac{\Delta t}{\Delta x} \right| < 1 \;\rightarrow\; c_g < u \qquad \text{(Lax-Wendroff)}$$

or

$$\mu > 0 \quad \text{for} \quad \left| u\frac{\Delta t}{\Delta x} \right| < 1 \;\rightarrow\; c_g > u \qquad \text{(Beam-Warming)}$$

original PDE (linear advection equation): $\mu = 0 \rightarrow$ no dispersion!

$$\omega = u\,k \qquad\qquad c_p = u \qquad\qquad c_g = u$$

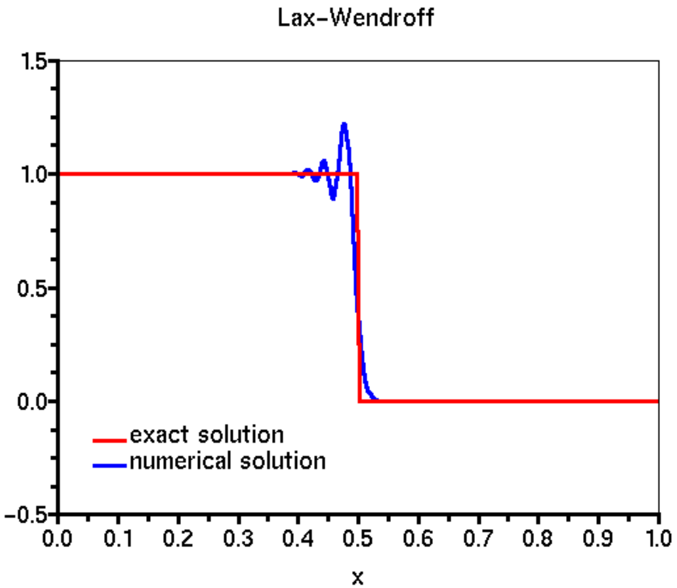
Analysis of Schemes – Summary

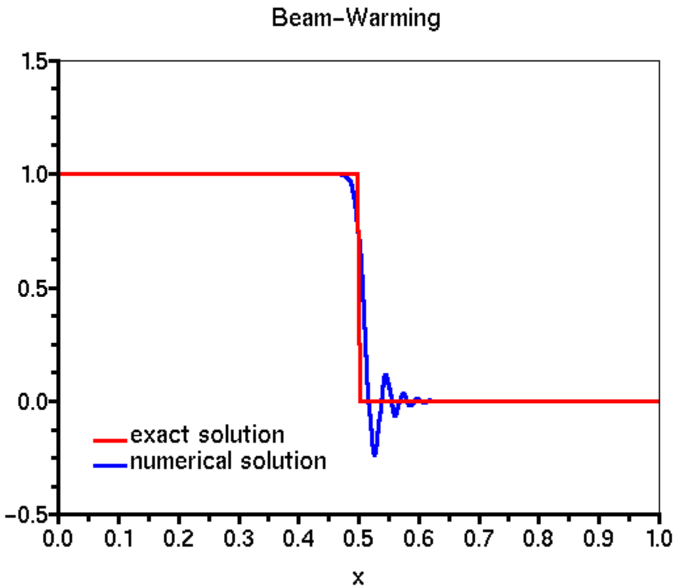
second order schemes:

modified equation: dispersive equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = \mu \frac{\partial^3 \rho}{\partial x^3}$$

$\rightarrow$ oscillating behaviour of the numerical solution





Analysis of Schemes – Summary

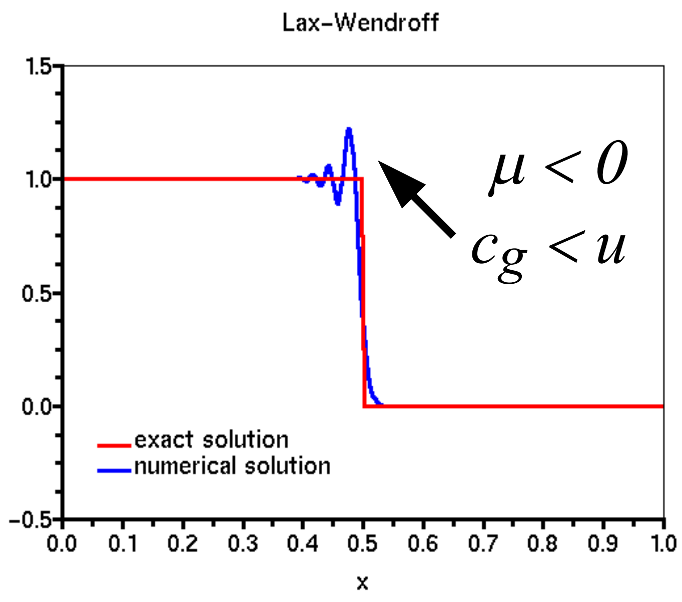
second order schemes:

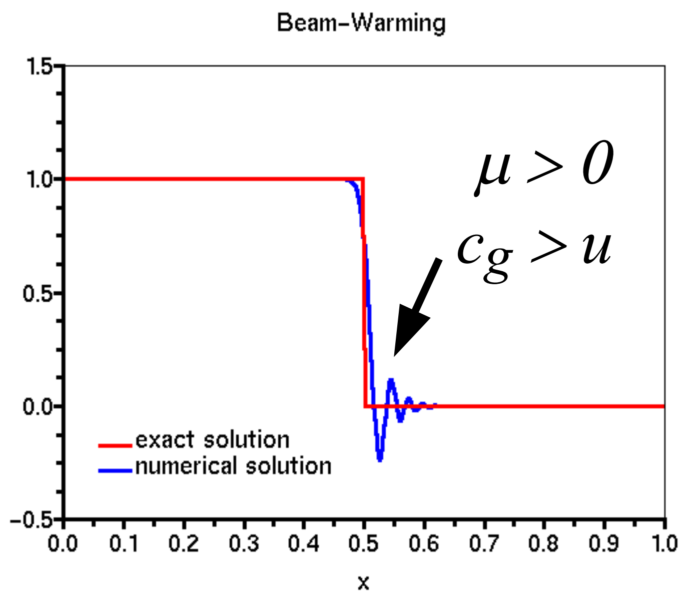
modified equation: dispersive equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = \mu \frac{\partial^3 \rho}{\partial x^3}$$

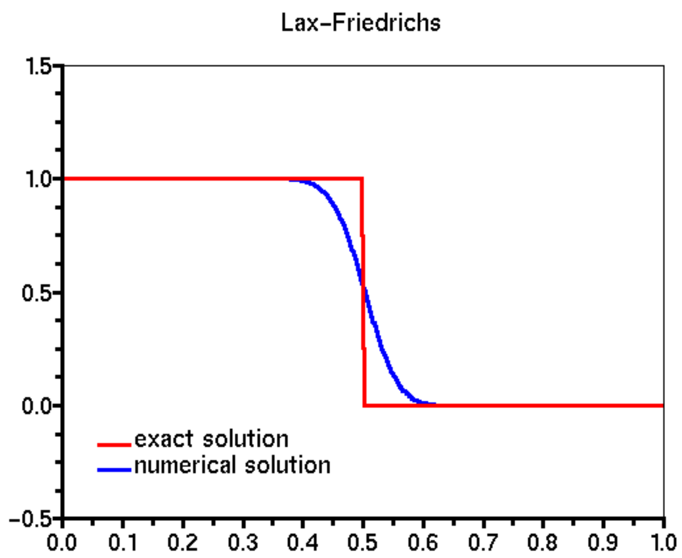
waves:
group velocity
$c_g = u + 3\mu k^2$

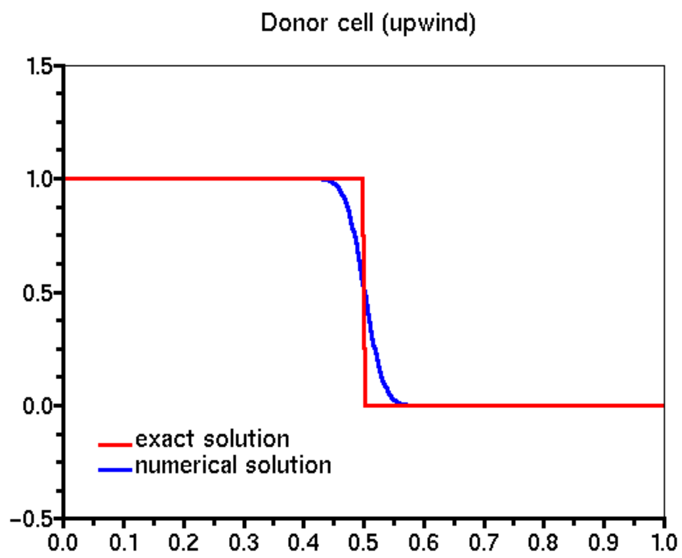
→ oscillating behaviour of the numerical solution





Analysis of Schemes – Summary





first order schemes:

modified equation: advection-diffusion equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = D \frac{\partial^2 \rho}{\partial x^2}$$

$\rightarrow$ diffusive behaviour of the numerical solution

Linear Stability Analysis – Original PDE

Lets see what happens to waves in the linear advection equation
For the ansatz

$$\rho(x,t) = A(t)\, e^{-jkx}$$

with $j^2 = -1$ we get

$$\frac{\mathrm{d}A}{\mathrm{d}t} + v\,(-jk)\,A = 0 \quad \Rightarrow \quad \frac{\mathrm{d}A}{\mathrm{d}t} = jvk\,A \quad \Rightarrow \quad A = A_0\, e^{jvkt} \quad ,$$

$$\rho = A_0\, e^{j(\omega t - kx)}$$

with

$$\mathrm{abs}(A) = \mathrm{abs}(A_0) = \mathrm{const} \quad ,$$

$$\omega = vk \quad .$$

original PDE (linear advection equation):

- constant amplitude (no growth or decay)
- no dispersion (all waves travel at same speed)

Linear Stability Analysis – General Idea

Linear stability analysis tells about stability - of linear schemes for linear equations.

Linear stability $\Rightarrow$ convergence for consistent schemes

Ansatz: for

$$\rho_i^n = e^{-jik\Delta x} \qquad \text{with } j^2 = -1$$

with $k \leq k_0 = \frac{\pi}{\Delta x}$ we search $A \in \mathbb{C}$ with

$$\rho_i^{n+1} = A\,\rho_i^n = A\,e^{-jik\Delta x} \;.$$

Amount $\mathrm{abs}(A)$ of A $\Rightarrow$ damping (diffusion) or growth (instability) of waves.

Phase of A $\Rightarrow$ wave speed and dispersion.

Ideally $A = e^{j\omega\Delta t}$ with $\omega = vk$.

Linear Stability Analysis – FTCS Scheme

Applying ansatz $\rho_i^n = e^{-jik\Delta x}$

$$A\, e^{-jik\Delta x} = e^{-jik\Delta x} - \frac{\Delta t}{\Delta x}\, v\, \frac{1}{2} \left(e^{-j(i+1)k\Delta x} - e^{-j(i-1)k\Delta x} \right) \quad .$$

Multiplying with $e^{jik\Delta x}$ and using the Courant number

$$\alpha := \frac{\Delta t}{\Delta x}\, v$$

we get

$$A = 1 - \alpha\, \frac{1}{2} \left(e^{-jk\Delta x} - e^{jk\Delta x} \right) = 1 + j\alpha \sin k\Delta x \quad ,$$

$$\mathrm{abs}(A) = \left(1 + \alpha^2 \sin^2 k\Delta x \right)^{\frac{1}{2}} \quad ,$$

$$\mathrm{abs}(A) > 1 \quad \text{for } \; 0 < k\Delta x < \pi \;\; , \;\; \alpha > 0 \;\; .$$

All waves except the ones with smallest wavenumber grow exponentially in time:
The scheme is unconditionally unstable, independent of the time-step.

Linear Stability Analysis – Donor Cell Scheme

Applying ansatz $\rho_i^n = e^{-jik\Delta x}$

$$A = 1 - \alpha\left(1 - e^{jk\Delta x}\right) = 1 - \alpha\left(1 - \cos k\Delta x\right) + j\alpha \sin k\Delta x \quad ,$$

$$\mathrm{abs}(A) = [1 - 2 \underbrace{\left(\alpha - \alpha^2\right)}_{\geq 0 \text{ for } \alpha\in[0,1]} \underbrace{\left(1 - \cos k\Delta x\right)}_{\geq 0}]^{\frac{1}{2}} \quad ,$$

$$\mathrm{abs}(A) \leq 1 \quad \text{for} \;\; \alpha \in [0,1] \quad .$$

The donor cell scheme is stable if the CFL condition is fulfilled,

$$\frac{\Delta t}{\Delta x} v \in [0,1] \quad .$$

Note: $v \geq 0$ is required (for $v \leq 0$ use FTFS).

Note: $\mathrm{abs}(A) < 1$ is possible: numerical viscosity

Note: $\mathrm{phase}(A) \neq vk\Delta t$: dispersion

Linear Stability Analysis – Remarks

- Severe restriction: linear PDE and linear scheme
- Growth of amplitude means instability - and stability otherwise
- Correct solution (constant amplitude) right at the border to instability
- Decline of amplitude indicates numerical viscosity
- Constant amplitude achievable by time-symmetric schemes (but: wiggles, dispersion)

Conclusions:

- No linear scheme is really satisfying.