Introduction to Numerical Hydrodynamics and Radiative Transfer

Part II: Hydrodynamics, Lecture 5

HT 2012

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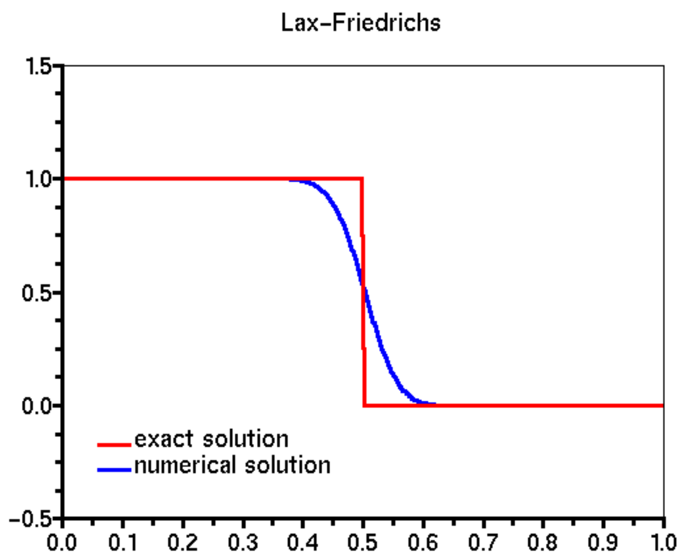
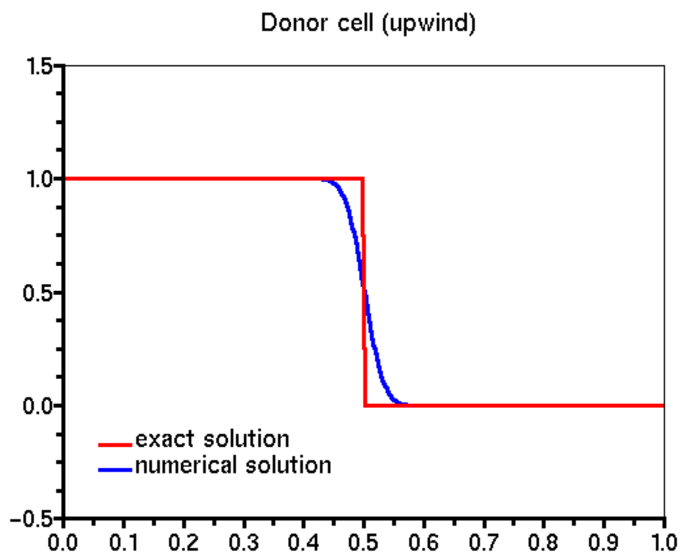
Introduction to Numerical Hydrodynamics

2. The Linear Advection Equation

2.6 PLM and Non-linear Schemes

Linear Advection – The Story So Far ...





first order schemes:

modified equation: advection-diffusion equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = D \frac{\partial^2 \rho}{\partial x^2}$$

→ diffusive behaviour of the numerical solution

Linear Advection – The Story So Far ...

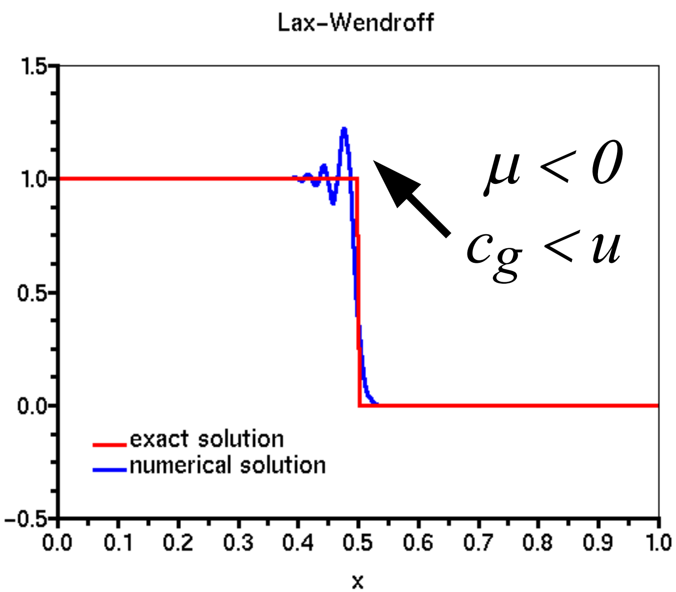
second order schemes:

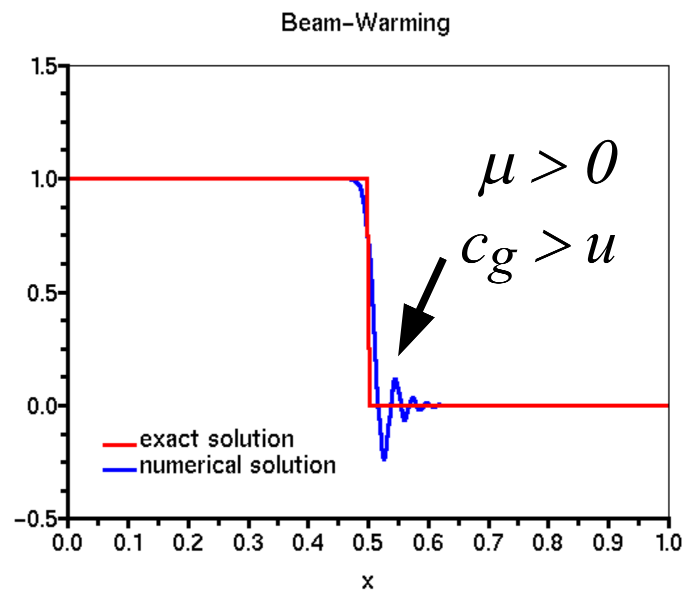
modified equation: dispersive equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = \mu \frac{\partial^3 \rho}{\partial x^3}$$

waves:
group velocity
$c_g = u + 3\mu k^2$

→ oscillating behaviour of the numerical solution





Numerical Schemes ... Alternatives?

Question:

How to construct a numerical scheme for advection which is more accurate than a first-order scheme (less diffusion, sharper gradients), but which doesn't develop oscillations like the second order schemes which we have investigated so far?

Let's start with recalling a few important concepts ...

Integral Form and Flux Centering

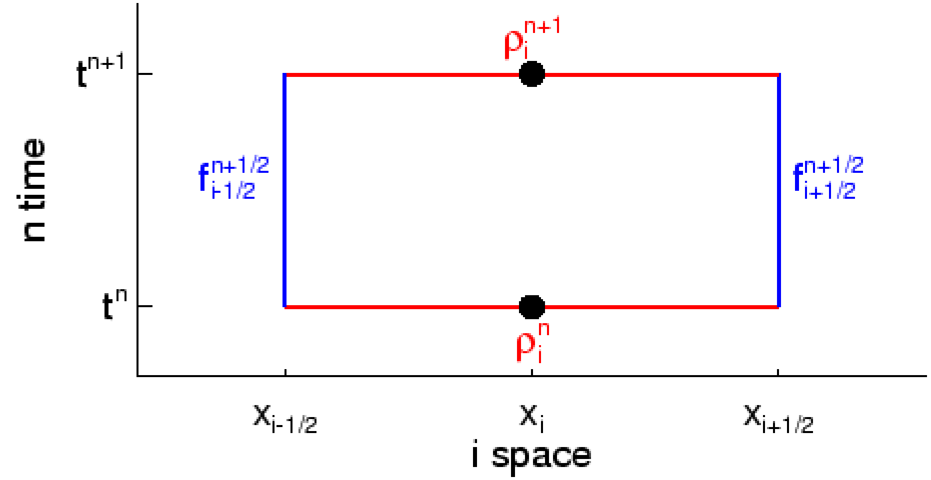


Figure courtesy of Bernd Freytag

Transformation: linear advection equation in integral form:

$$\int_{x_0}^{x_1} [\rho(x,t_1) - \rho(x,t_0)]\, dx + v \int_{t_0}^{t_1} [\rho(x_1,t) - \rho(x_0,t)]\, dt = 0$$

... for one grid cell and one time step:

$$\underbrace{\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x,t^{n+1})\, dx}_{\Delta x\, \rho_i^{n+1}} - \underbrace{\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \rho(x,t^{n})\, dx}_{\Delta x\, \rho_i^{n}} + \underbrace{\int_{t^n}^{t^{n+1}} v\, \rho\Big(x_{i+\frac{1}{2}},t\Big)\, dt}_{\Delta t\, f_{i+\frac{1}{2}}^{n+\frac{1}{2}}} - \underbrace{\int_{t^n}^{t^{n+1}} v\, \rho\Big(x_{i-\frac{1}{2}},t\Big)\, dt}_{\Delta t\, f_{i-\frac{1}{2}}^{n+\frac{1}{2}}} = 0$$

Update Formula in Conservation Form

After computing the fluxes at the cell boundaries $f^n_{i+\frac{1}{2}}$ that characterize a method

(e.g. from the fluxes in the cells: $f(\rho_i^n) = v\,\rho_i^n$)

the update can be done by the formula

$$\rho_i^{n+1} = \rho_i^n - \frac{\Delta t}{\Delta x}\left(f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}}\right) \quad .$$

This is the conservation form because the density changes only due to fluxes through the boundaries, and is conserved otherwise:

$$\begin{aligned}
\sum_{i=i_0}^{i_1} \rho_i^{n+1} &= \sum_{i=i_0}^{i_1} \rho_i^n + \frac{\Delta t}{\Delta x} \sum_{i=i_0}^{i_1} \left(f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}}\right) \\
&= \sum_{i=i_0}^{i_1} \rho_i^n + \frac{\Delta t}{\Delta x} \left[f^n_{i_1+\frac{1}{2}} + \sum_{i=i_0}^{i_1-1} \left(f^n_{i+\frac{1}{2}} - f^n_{i+\frac{1}{2}}\right) - f^n_{i_0-\frac{1}{2}}\right] \\
&= \sum_{i=i_0}^{i_1} \rho_i^n + \frac{\Delta t}{\Delta x} \left(f^n_{i_1+\frac{1}{2}} - f^n_{i_0-\frac{1}{2}}\right) \quad .
\end{aligned}$$

Update Formula in Conservation Form

After computing the fluxes at the cell boundaries $f^n_{i+\frac{1}{2}}$ that characterize a method

(e.g. from the fluxes in the cells: $f(\rho^n_i) = v\,\rho^n_i$)

the update can be done by the formula

$$\rho^{n+1}_i = \rho^n_i - \frac{\Delta t}{\Delta x}\left(f^n_{i+\frac{1}{2}} - f^n_{i-\frac{1}{2}}\right) \quad .$$

This is the conservation form because the density changes only due to fluxes through the boundaries, and is conserved otherwise.

Guidelines for improved advection:

- keep conservation form
- define more accurate fluxes (reduce diffusion or oscillations)

Improving Advection – RSA Algorithm

Reconstruct – Solve – Average (RSA):

Godunov-type finite-volume scheme

1. Reconstruct $\rho(x)$ from the discrete values ρ_i^n (cell averages)
2. Solve the exact problem for Δt
3. Average this function over each grid cell to obtain ρ_i^{n+1}

Steps 2 and 3 are well-behaved (e.g., conservative) by default, step 1 is potentially a problem.

Piecewise constant reconstruction (constant value in cell)
$\rightarrow$ Godunov's method

Improving Advection – PLM Schemes

- Donor cell scheme: constant ρ within each cell
- Improvement: piecewise linear method (PLM): reconstruction by linear function in each cell

Once we know the cell average ρ_i and the slope $\delta\rho_i$, we get the flux over Δt from

$$f_{i\pm\frac{1}{2}} = v\left\{\rho_i + \frac{1}{2}\,\delta\rho_i\,\mathrm{sign}(v)\left[1 - \mathrm{abs}(v)\,\frac{\Delta t}{\Delta x}\right]\right\} \quad .$$

If the boundary value in the cell is used for the entire cell we get

$$f_{i\pm\frac{1}{2}} = v\left\{\rho_i + \frac{1}{2}\,\delta\rho_i\,\mathrm{sign}(v)\right\} \quad .$$

For $v > 0$ we get from each cell ρ_i the flux $f_{i+\frac{1}{2}}$.

For $v < 0$ we get from each cell ρ_i the flux $f_{i-\frac{1}{2}}$.

PLM Schemes – Examples of Slopes

Slopes of already encountered (linear) schemes:
Donor cell scheme (slope zero):

$$\delta\rho_i = 0$$

Lax-Wendroff scheme:

$$\delta\rho_i = \rho_{i+1} - \rho_i$$

Beam-Warming scheme:

$$\delta\rho_i = \rho_i - \rho_{i-1}$$

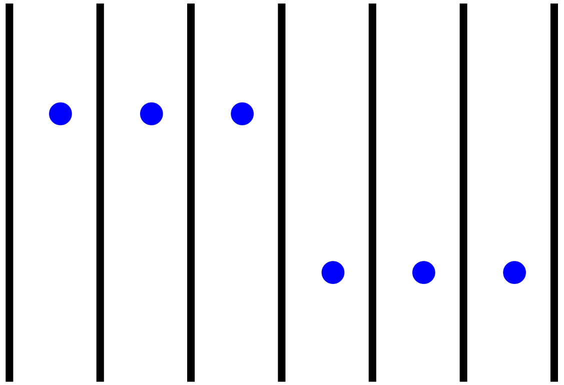
Fromm scheme:

$$\delta\rho_i = \frac{1}{2}\left(\rho_{i+1} - \rho_{i-1}\right)$$

PLM Schemes – Examples of Slopes

Example: Lax-Wendroff ... the RSA perspective

... applied to piecewise constant data •



PLM Schemes – Examples of Slopes

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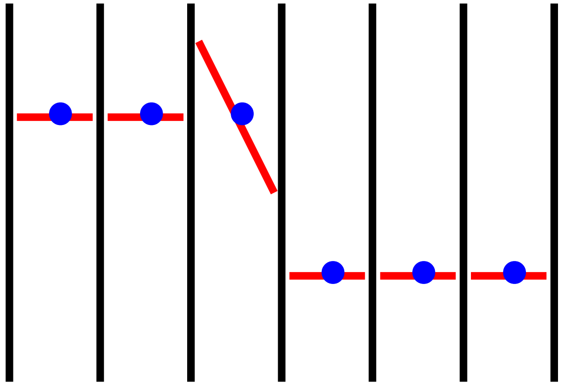

reconstruction

PLM Schemes – Examples of Slopes

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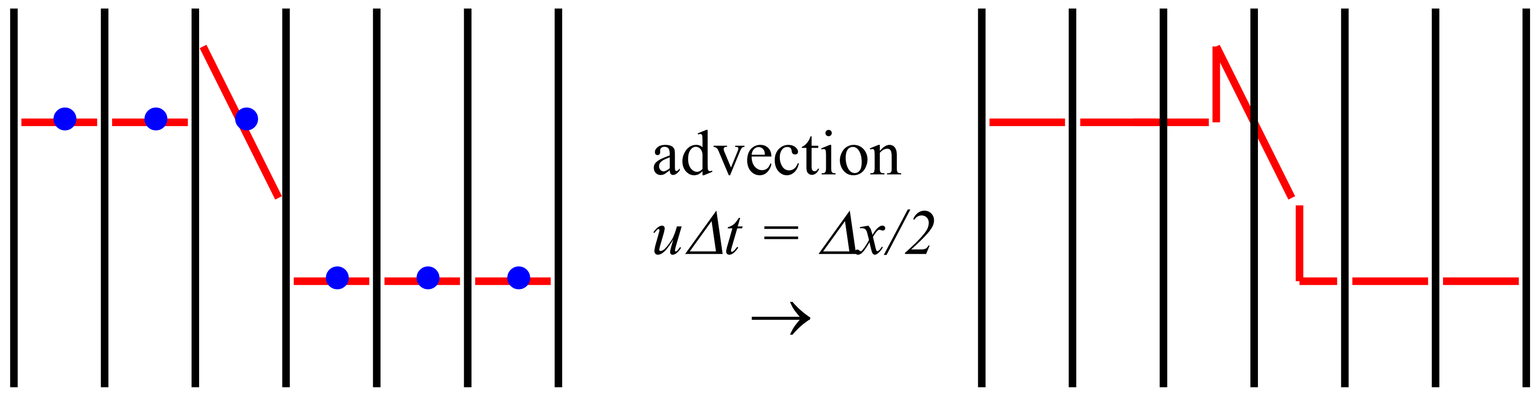

reconstruction

PLM Schemes – Examples of Slopes

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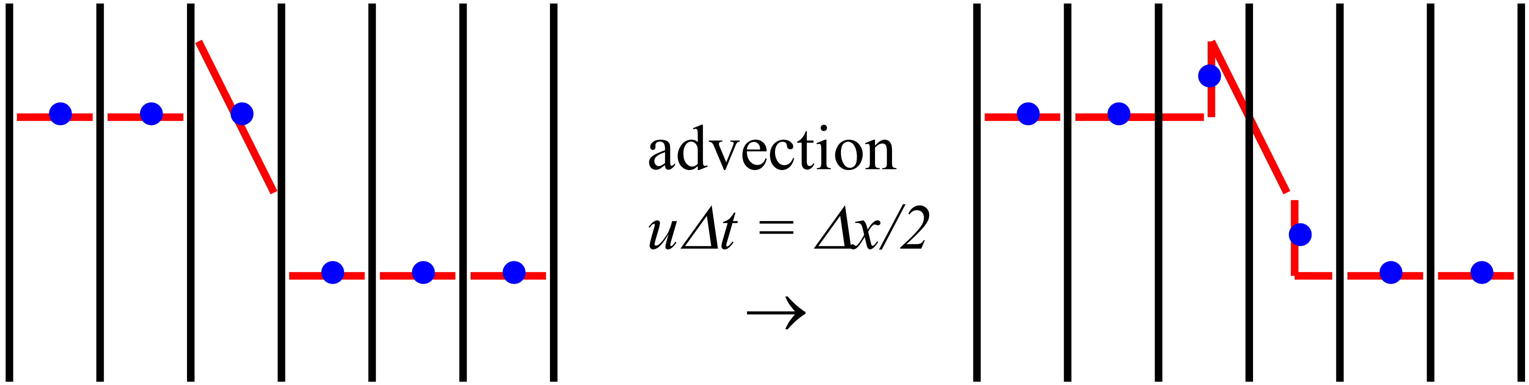


reconstruction   new cell averages

$\rightarrow$ too steep slope causes oscillations ...

Improving Advection – Monotonicity

Transport and averaging are easy and well determined. The entire algorithm is determined by the reconstruction scheme.

- Consistency: ``reasonable'' interpolation
- Accuracy: high-order polynomials
- Conservativity: proper flux formulation
- Stability, positivity: monotonicity preservation

Definition: Total variation:

$$TV = \sum_i \mathrm{abs}(\rho_{i+1} - \rho_i)$$

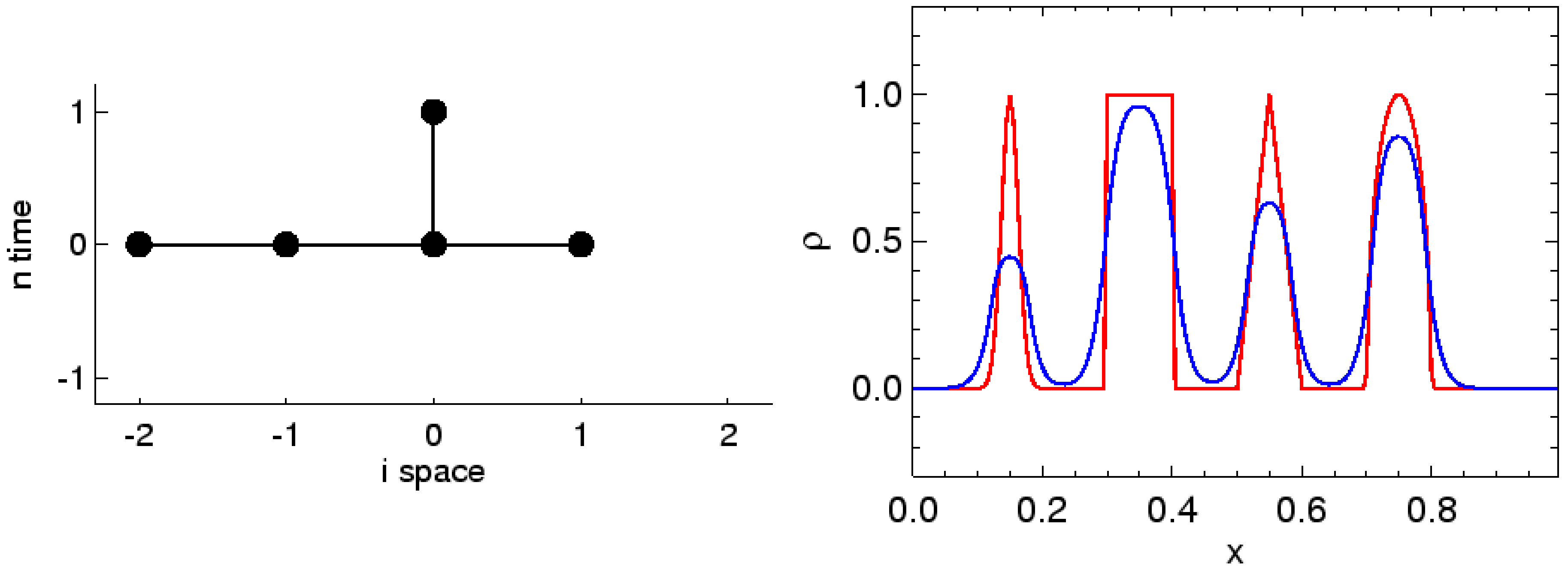
TVD property: A scheme has the TVD property (is ``Total Variation Diminishing'') if TV does not increase in time.

$\Rightarrow$ Non-linear stability criterion.

Godunov's theorem: linear monotonicity-preserving methods are first-order accurate, at best.

$\Rightarrow$ Try a non-linear scheme.

PLM Scheme with Minmod Slope-Limiter

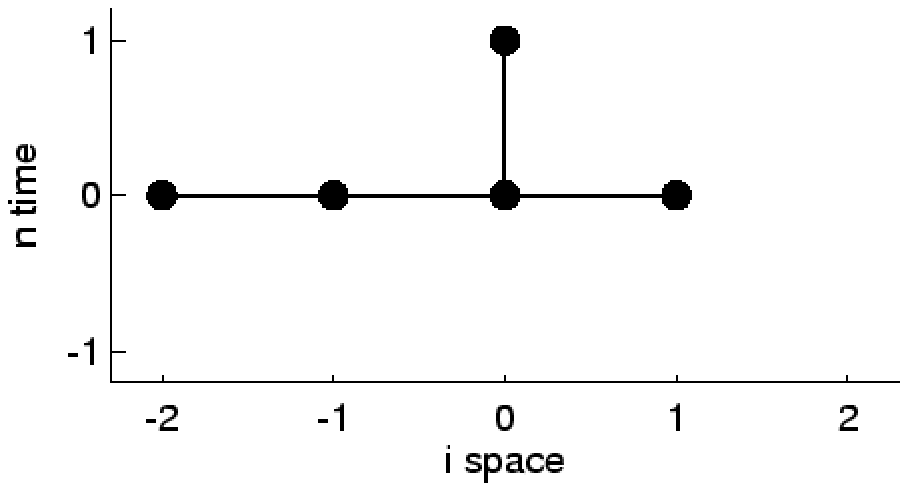


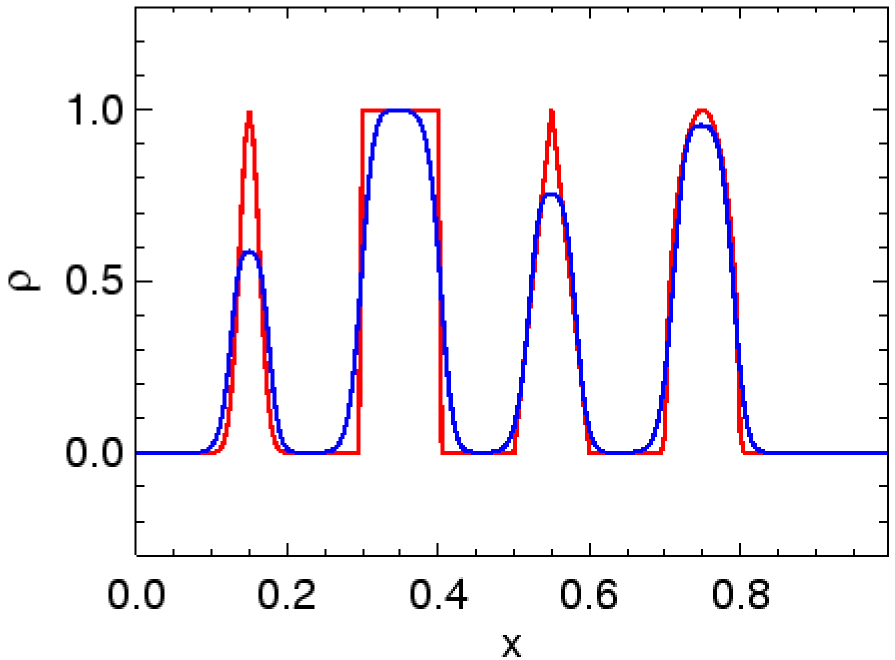
Figures courtesy of Bernd Freytag

Stencil diagram and test result (initial condition: red, solution: blue) for PLM scheme with minmod slope-limiter

$$\delta\rho_i = \min(\max(\rho_i - \rho_{i-1}, 0), \max(\rho_{i+1} - \rho_i, 0)) + \max(\min(\rho_i - \rho_{i-1}, 0), \min(\rho_{i+1} - \rho_i, 0)) \quad .$$

PLM Scheme with van Leer Slope-Limiter





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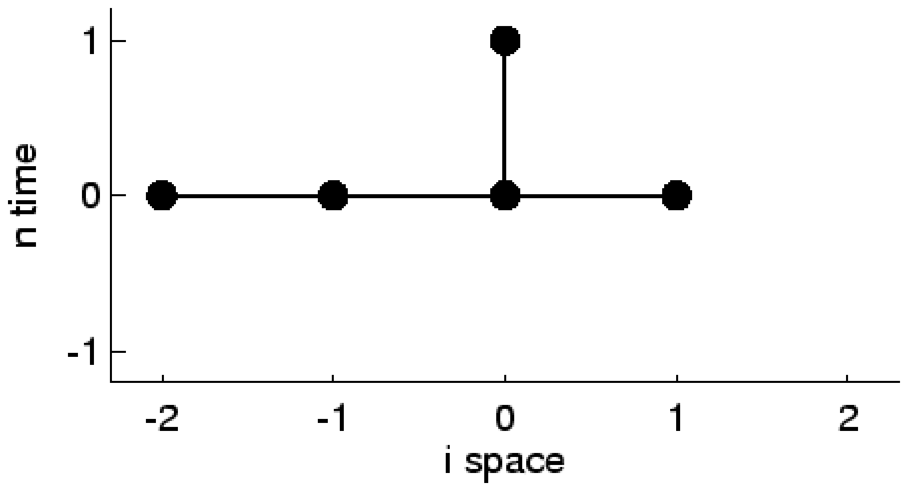
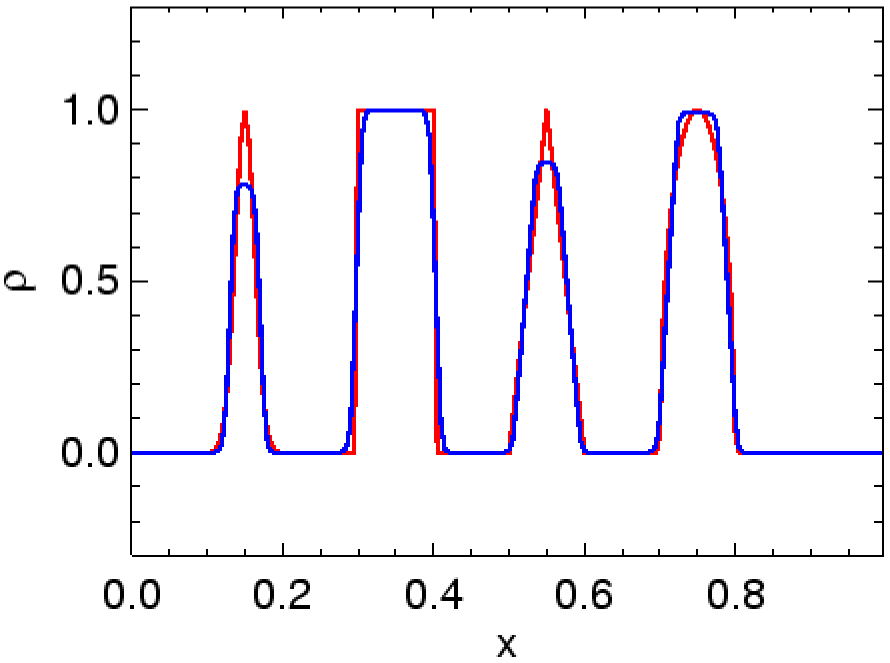
Stencil diagram and test result (initial condition: red, solution: blue) for PLM scheme with van Leer slope-limiter (harmonic mean of slopes)

$$\delta\rho_i = \begin{cases} \dfrac{1}{\frac{1}{2}\left(\frac{1}{\rho_i - \rho_{i-1}} + \frac{1}{\rho_{i+1} - \rho_i}\right)} & \text{if} \quad (\rho_i - \rho_{i-1})(\rho_{i+1} - \rho_i) > 0 \\ 0 & \text{elsewhere} \end{cases} .$$

PLM Scheme with Superbee Slope-Limiter





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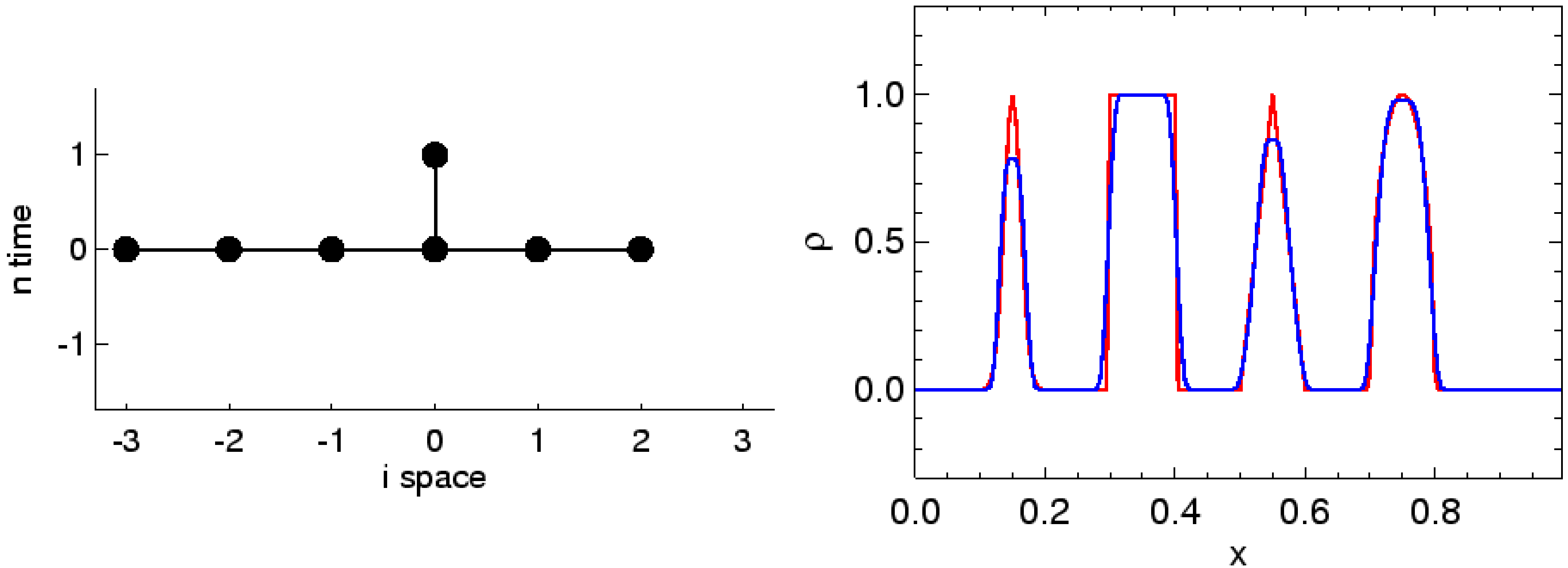
Stencil diagram and test result (initial condition: red, solution: blue) for PLM scheme with Superbee slope-limiter

$$\delta\rho_i = [\mathrm{sign}(\rho_i - \rho_{i-1}) + \mathrm{sign}(\rho_{i+1} - \rho_i)]$$

$$\min\left(\mathrm{abs}(\rho_i - \rho_{i-1})\,,\mathrm{abs}(\rho_{i+1} - \rho_i)\,,\frac{1}{2}\max(\mathrm{abs}(\rho_i - \rho_{i-1})\,,\mathrm{abs}(\rho_{i+1} - \rho_i))\right)$$

PPM Scheme



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Stencil diagram and test result (initial condition: red, solution: blue) for PPM scheme with piecewise parabolic reconstruction (Colella & Woodward, 1984)