Introduction to Numerical Hydrodynamics and Radiative Transfer

Part II: Hydrodynamics, Lecture 6

HT 2012

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Introduction to Numerical Hydrodynamics

3. Non-linear Advection

3.1 Introduction of Burgers' Equation

Burgers' Equation and the Euler Equations

The Euler equations, 1D:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho u\right) + \frac{\partial}{\partial x}\left(\rho u u + P\right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \frac{\partial}{\partial x}\left((\frac{1}{2}\rho u^2 + \rho e + P)u\right) = 0$$

Burgers' Equation and the Euler Equations

The Euler equations, 1D:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

reformulate the equation of motion:

$$\boxed{\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho u u + P) = 0}$$

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$$u\frac{\partial \rho}{\partial t} + u\frac{\partial}{\partial x}\left(\rho u\right) + \rho\frac{\partial u}{\partial t} + \left(\rho u\right)\frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0$$

Burgers' Equation and the Euler Equations

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$$\underbrace{u\frac{\partial \rho}{\partial t} + u\frac{\partial}{\partial x}(\rho u)}_{=0} + \rho\frac{\partial u}{\partial t} + (\rho u)\frac{\partial u}{\partial x} + \frac{\partial P}{\partial x} = 0$$

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... and neglect pressure gradients:

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... and neglect pressure gradients:

Burgers' equation captures the essential non-linearity of the 1D Euler equation of motion.

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0}$$

Burgers' Equation: Definitions

inviscid Burgers' Equation:

$$\frac{\partial v}{\partial t} + \frac{\partial \frac{1}{2} v^2}{\partial x} = 0$$

in conservation form (flux $\frac{1}{2} v^2$)

... in quasi-linear form: $$\boxed{\frac{\partial v}{\partial t} + v\,\frac{\partial v}{\partial x} = 0}$$

characteristic curves *c(t)*:

$$\frac{\mathrm{d}c}{\mathrm{d}t} = v \qquad \Longrightarrow \qquad c(t) = v(x_0)\; t + x_0$$

velocity v is constant along characteristics
(proof by checking, *dv/dt = 0*)

$\Rightarrow$ 'graphical solution'

Burgers' Equation: Solutions

inviscid Burgers' Equation:

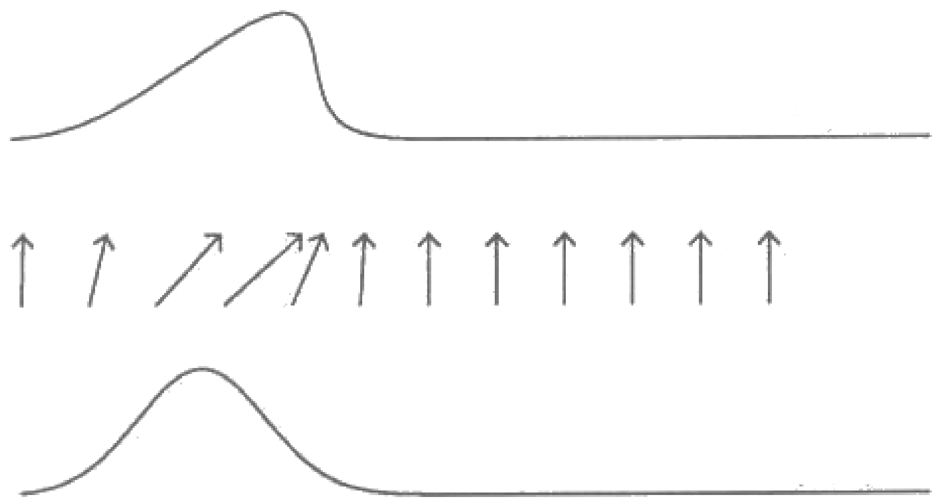
solution $v(x,t)$

$\Uparrow$

characteristics $c(t)$

$\Uparrow$

initial data $v(x_0)$



Characteristics and solution for Burgers' equation (small t).

$$\frac{\mathrm{d}c}{\mathrm{d}t} = v \qquad \Rightarrow \qquad c(t) = v(x_0)\; t + x_0$$

velocity v is constant along characteristics

$\Rightarrow$ 'graphical solution'

Burgers' Equation: Solutions

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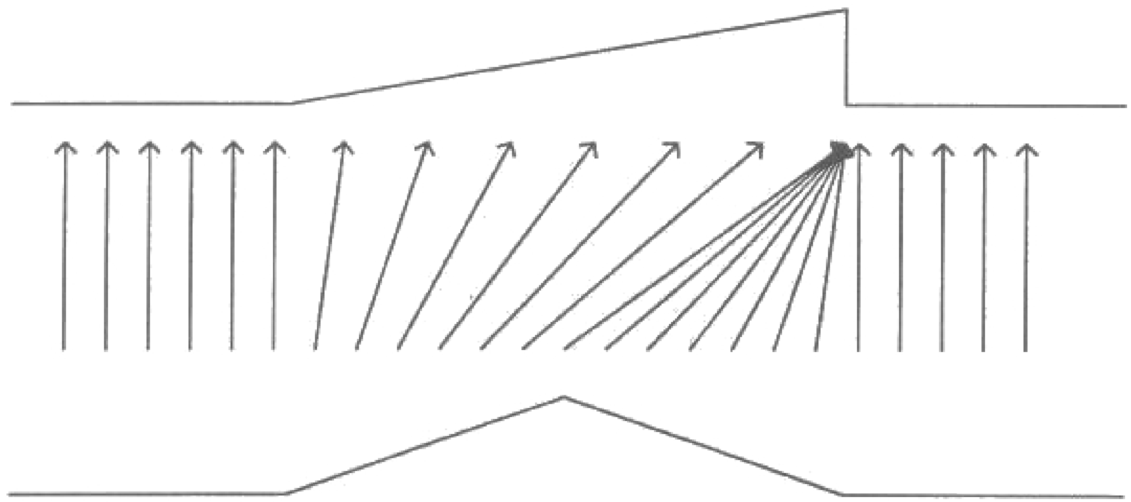

Shock formation in Burgers' equation.

$$\frac{\mathrm{d}c}{\mathrm{d}t} = v \qquad \Rightarrow \qquad c(t) = v(x_0)\ t + x_0$$

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Burgers' Equation: Solutions

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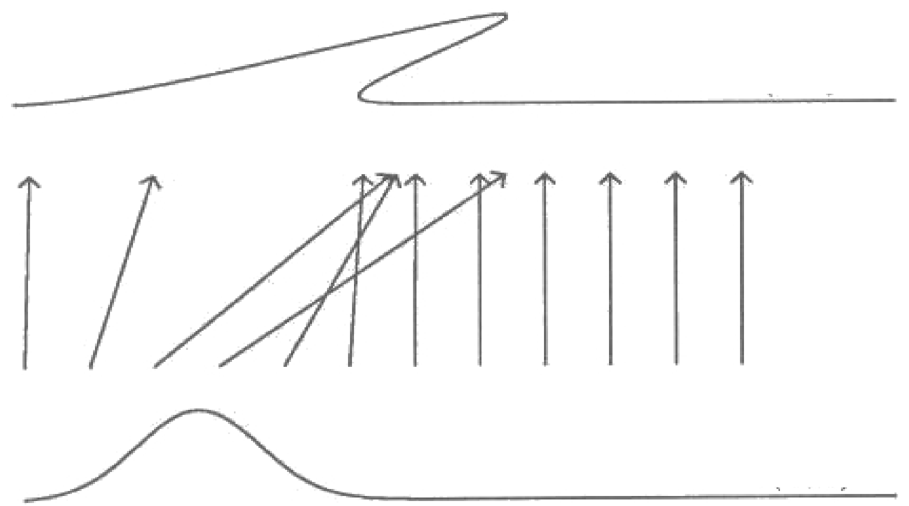


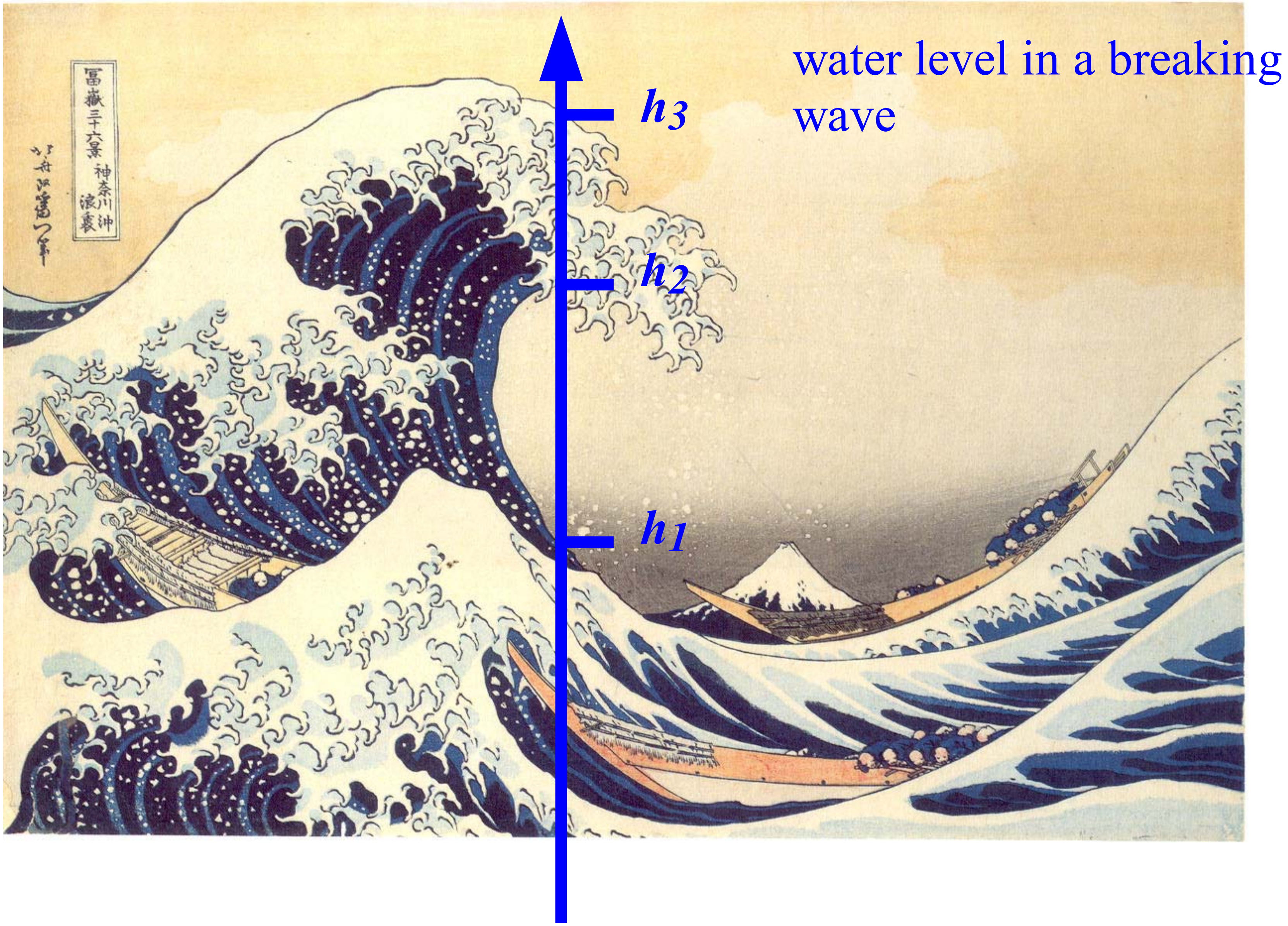
Triple-valued solution to Burgers' equation at time $t > T_b$.

$$\frac{\mathrm{d}c}{\mathrm{d}t} = v \qquad \Longrightarrow \qquad c(t) = v(x_0)\, t + x_0$$

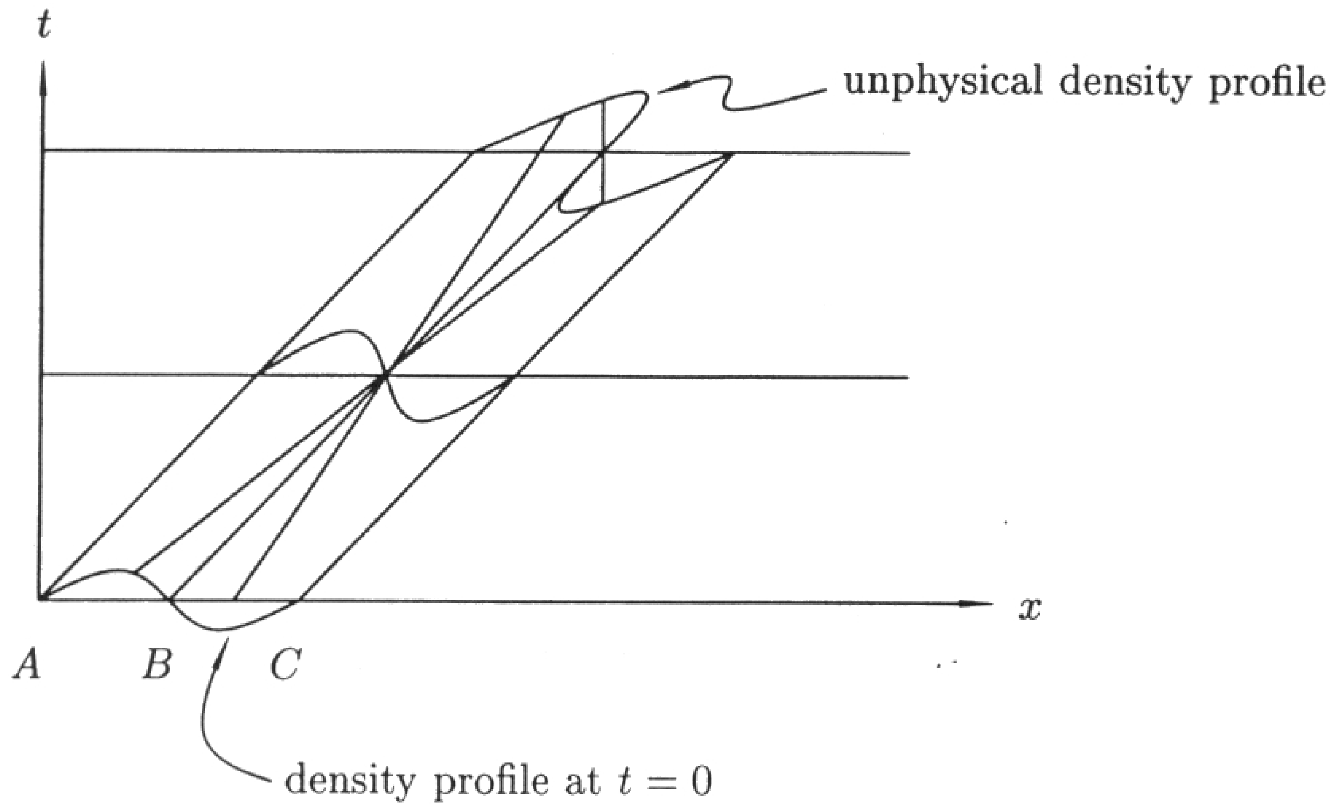
velocity v is constant along characteristics

$\Rightarrow$ 'graphical solution'

Triple-valued solutions that make sense ...

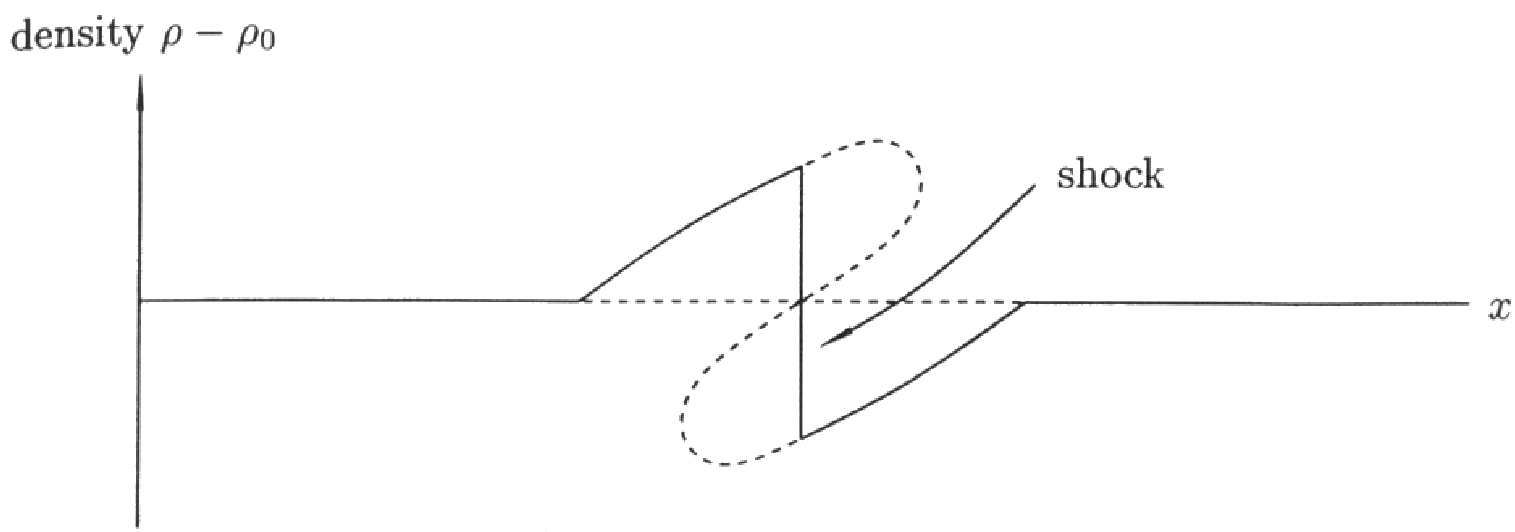


Triple-valued solutions that don't make sense ...



An acoustic wave of finite amplitude, even if it starts with a perfect sinusoidal shape and propagates in an undisturbed medium of exactly uniform properties, would inevitably steepen in its waveform.

Triple-valued solutions that don't make sense ...



The tendency for nonlinearities to steepen the wave profile, which would produce multiple values for fluid properties such as gas density and velocity, must be eventually offset by the onset of strong viscous forces. The balance of the viscous forces and the steepening tendency mediates a shock, which is approximated in ideal fluid flow as a discontinuous jump of gas properties across the front.

Burgers' Equation: Viscosity

Viscous and inviscid Burgers' Equation:

$$\boxed{\frac{\partial v}{\partial t} + \frac{\partial \frac{1}{2} v^2}{\partial x} = \epsilon \frac{\partial^2 v}{\partial x^2}}$$

$$\frac{\partial v}{\partial t} + \frac{\partial \frac{1}{2} v^2}{\partial x} = 0$$

Solution to the viscous Burgers' equation for two different values of ϵ.

Goal: capture the vanishing viscosity solution by solving the inviscid equation ...

Burgers' Equation: Compression Waves

A compression wave with $\partial v/\partial x < 0$ steepens with time and characteristic curves can cross: multiple-valued solution?

The viscous Burgers equation is a parabolic PDE and has a unique solution for all times $t > 0$.

Vanishing viscosity: we search for a solution of the inviscid Burgers equation which is a solution of the viscous Burgers equation in the limit $\varepsilon \to 0$.

Instead of a multiple-valued solution we get a discontinuity where the characteristics end.

Discontinuities (shocks) are unavoidable.

Discontinuities should be allowed in the initial conditions.

→ Riemann problem: conservation law together with piecewise constant data with a single discontinuity.

Burgers' Equation: The Riemann Problem

piecewise constant
initial data:

$$v(x,0) = \begin{cases} v_l & x < 0 \\ v_r & x > 0 \end{cases}$$

Burgers' Equation: The Riemann Problem

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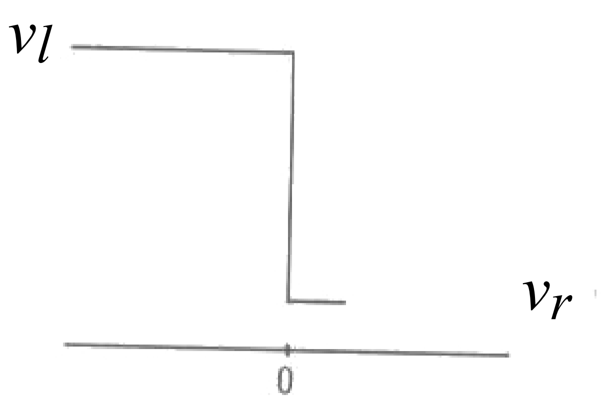
Case I:

$v_l > v_r \quad (> 0)$



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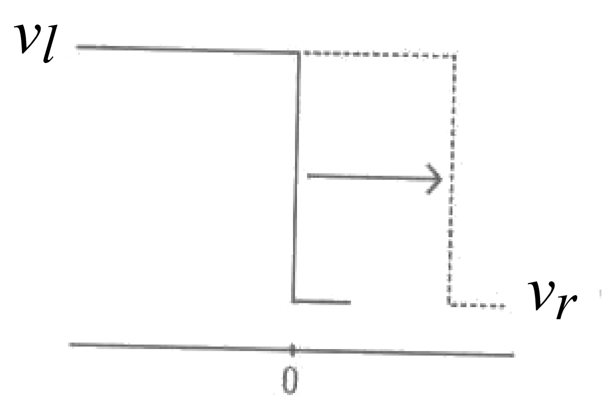
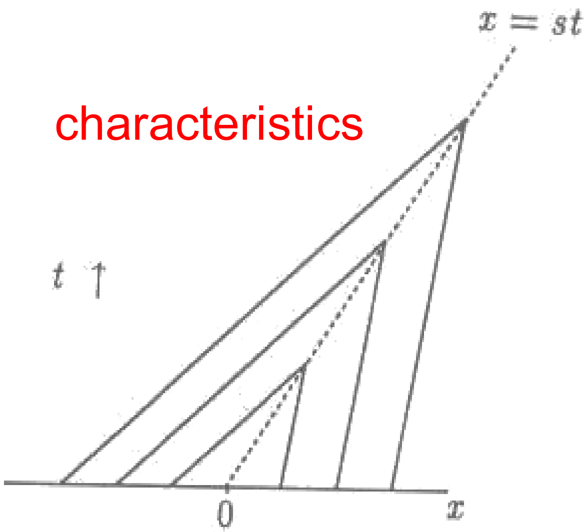


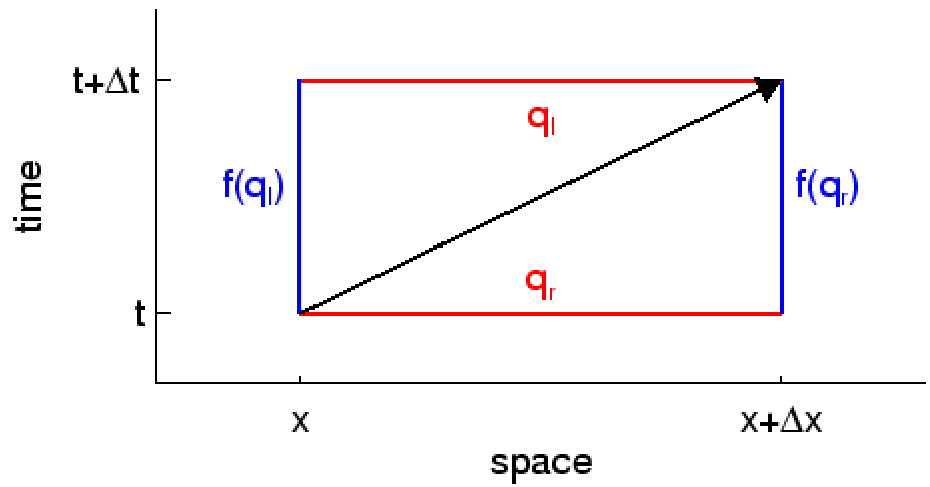


$\Rightarrow$ unique weak solution:
shock wave

$$v(x,t) = \begin{cases} v_l & x<st \\ v_r & x>st \end{cases}$$

The Riemann Problem: Shock Speed

In a time interval Δt
a shock with speed s travels
a distance $\Delta x = s\ \Delta t$.



Integration of the PDE

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0$$

over Δt and Δx results in

$$\int_x^{x+\Delta x} \left[q(x, t+\Delta t) - q(x,t)\right] dx + \int_t^{t+\Delta t} \left[f(q(x+\Delta x, t)) - f(q(x,t))\right] dt = 0$$

For almost constant states and fluxes to the left and right we get

$$\Delta x\, q_{\mathrm{l}} - \Delta x\, q_{\mathrm{r}} + \Delta t\, f(q_{\mathrm{r}}) - \Delta t\, f(q_{\mathrm{l}}) = O\big(\Delta t^2\big) \quad .$$

For $\Delta x = s\ \Delta t$ and $\Delta t \to 0$ we get the shock speed

$$s = \frac{f(q_{\mathrm{r}}) - f(q_{\mathrm{l}})}{q_{\mathrm{r}} - q_{\mathrm{l}}} \qquad \to \text{ for Burgers' equ.:} \qquad s = \frac{\frac{1}{2}v_{\mathrm{r}}^2 - \frac{1}{2}v_{\mathrm{l}}^2}{v_{\mathrm{r}} - v_{\mathrm{l}}} = \frac{1}{2}(v_{\mathrm{r}} + v_{\mathrm{l}})$$

Burgers' Equation: The Riemann Problem

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initial data:

$$v(x,0) = \begin{cases} v_l & x<0 \\ v_r & x>0 \end{cases}$$

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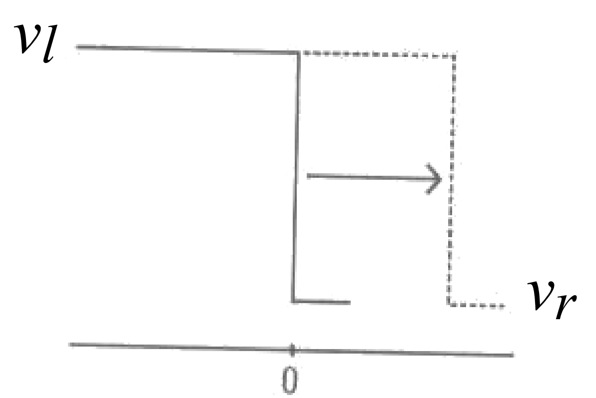
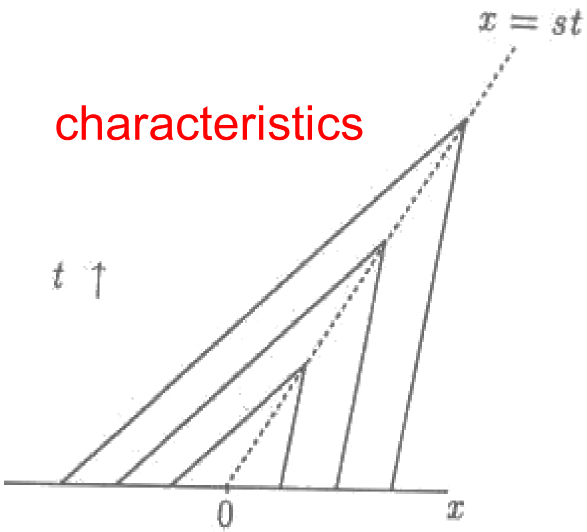


characteristics



$\Rightarrow$ unique weak solution:
shock wave

$$v(x,t) = \begin{cases} v_l & x<st \\ v_r & x>st \end{cases}$$

with shock speed

$$s = \frac{1}{2}(v_l + v_r)$$

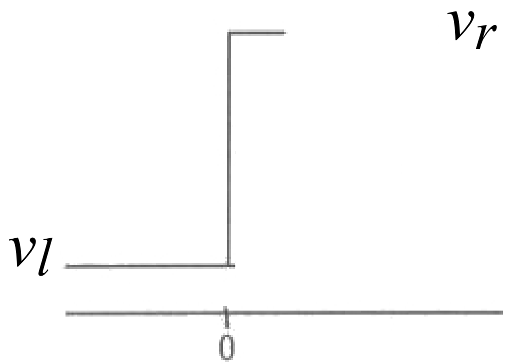
Burgers' Equation: The Riemann Problem

piecewise constant
initial data:

$$v(x,0) = \begin{cases} v_l & x<0 \\ v_r & x>0 \end{cases}$$

Case II:

$v_l < v_r \quad (> 0)$



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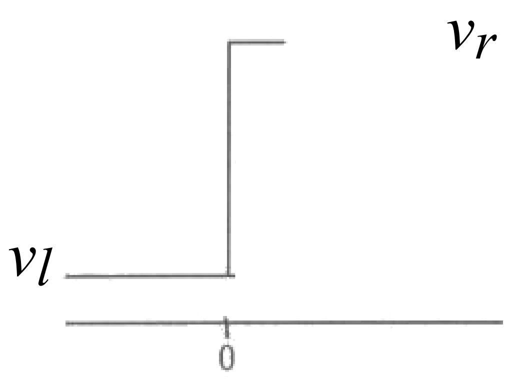

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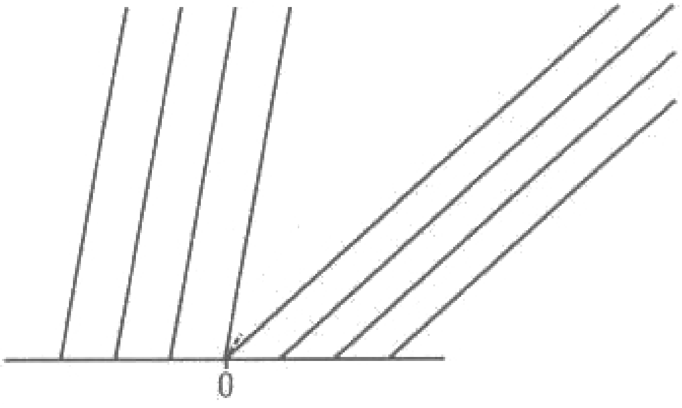

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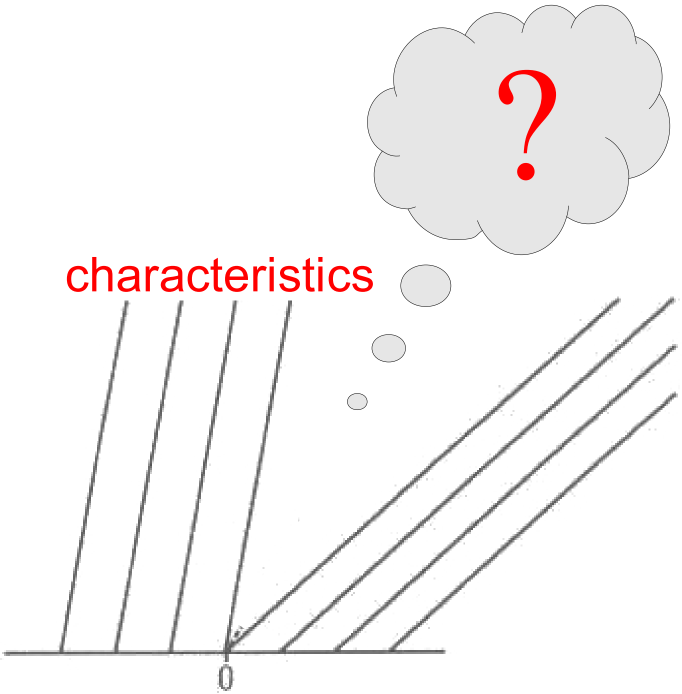

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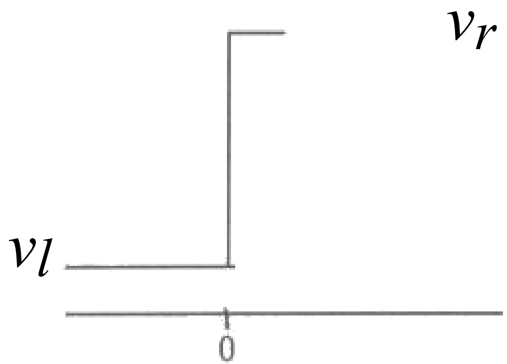

$\Rightarrow$ infinitely many (!)
weak solutions ...

Burgers' Equation: The Riemann Problem

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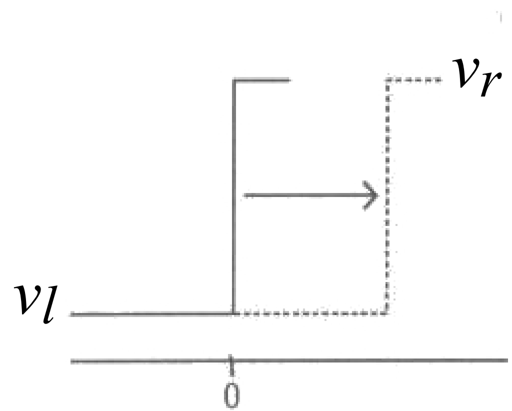
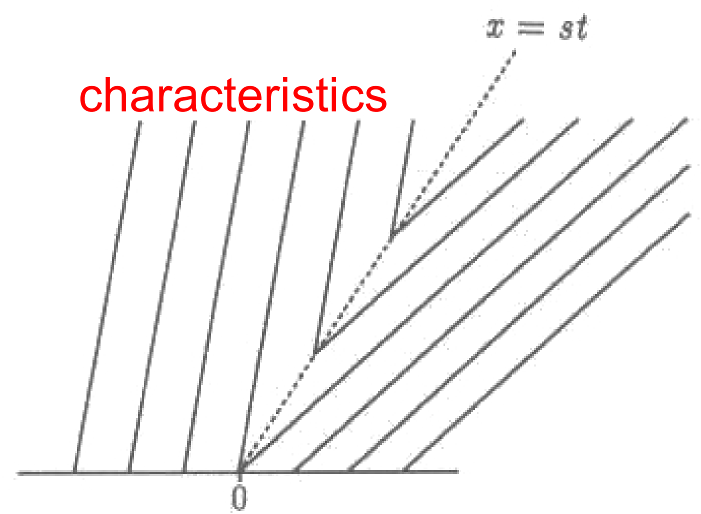




$\Rightarrow$ infinitely many (!)
weak solutions ...

$$v(x,t) = \begin{cases} v_l & x<st \\ v_r & x>st \end{cases}$$

with shock speed

$$s = \frac{1}{2}(v_l+v_r)$$

example:
entropy-
violating
shock

Burgers' Equation: The Riemann Problem

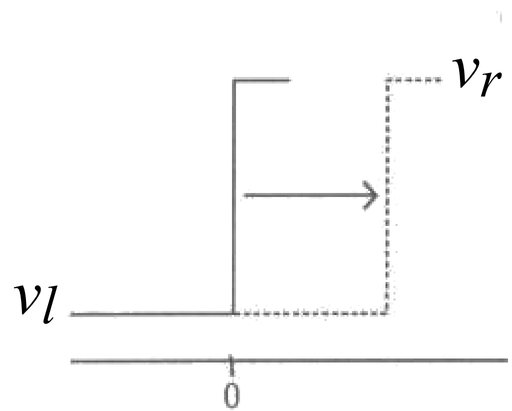
piecewise constant initial data:

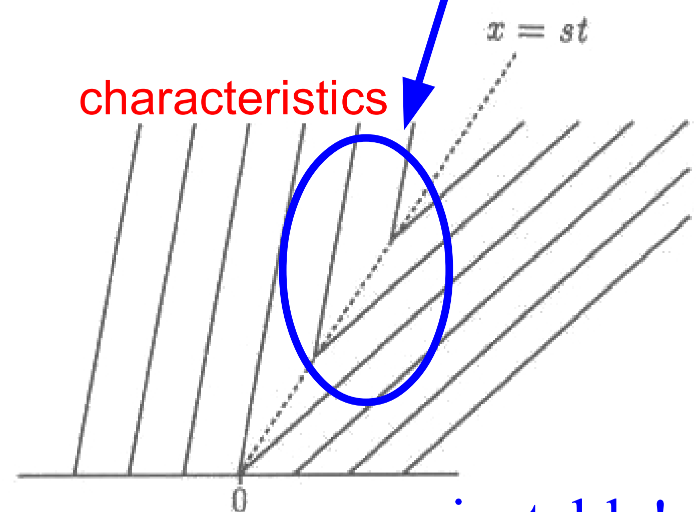
$$v(x,0) = \begin{cases} v_l & x<0 \\ v_r & x>0 \end{cases}$$

characteristics going out of the shock

Case II:

$v_l < v_r \quad (> 0)$





instable!

$\Rightarrow$ infinitely many (!) weak solutions ...

$$v(x,t) = \begin{cases} v_l & x<st \\ v_r & x>st \end{cases}$$

example: entropy-violating shock

with shock speed

$$s = \frac{1}{2}(v_l+v_r)$$

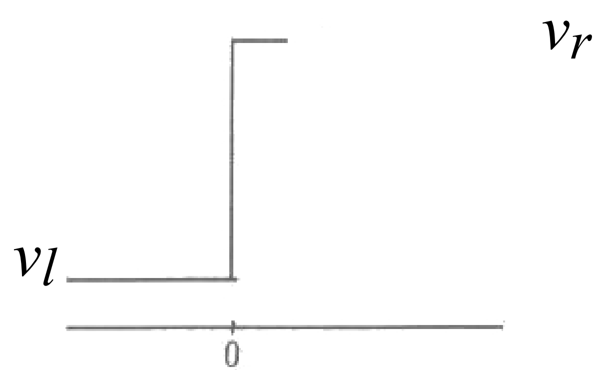
Burgers' Equation: The Riemann Problem

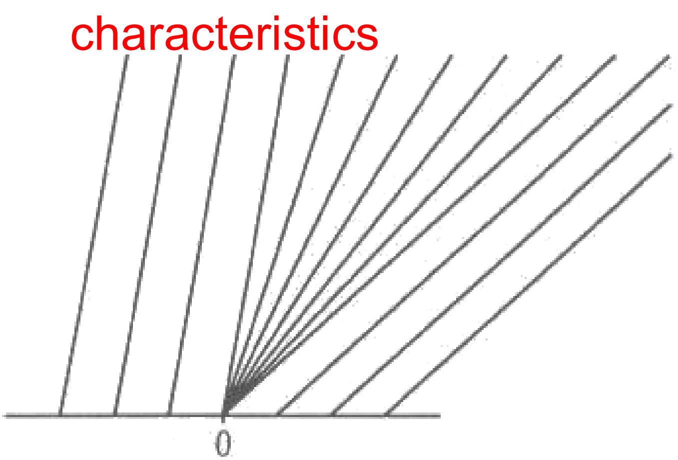
piecewise constant
initial data:

$$v(x,0) = \begin{cases} v_l & x<0 \\ v_r & x>0 \end{cases}$$

Case II:

$v_l < v_r \quad (> 0)$





$\Rightarrow$ infinitely many (!)
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Burgers' Equation: The Riemann Problem

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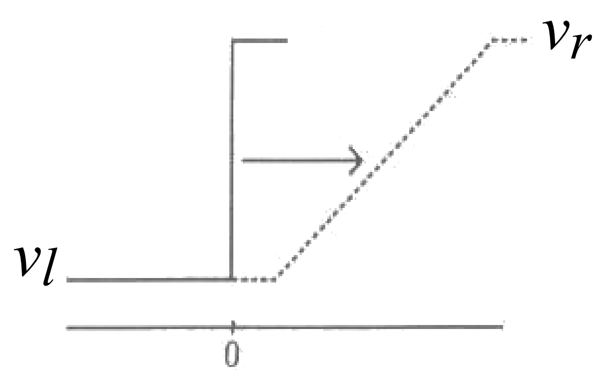
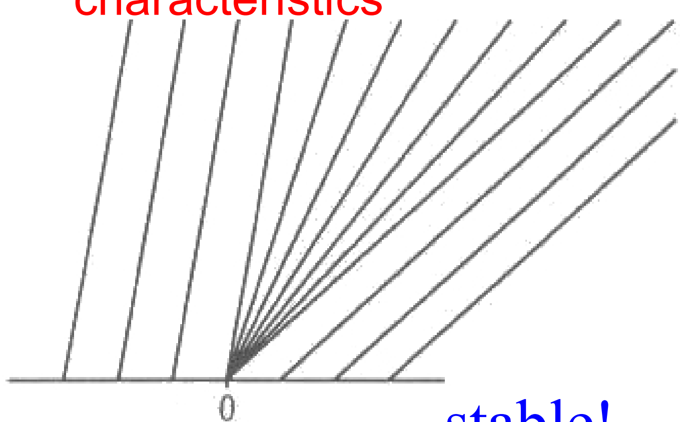




stable!

$\Rightarrow$ infinitely many (!)
weak solutions ...

$$v(x,t) = \begin{cases} v_l & x<v_l t \\ x/t & v_l t \le x \le v_r t \\ v_r & x>v_r t \end{cases}$$

example:
rarefaction
wave

vanishing viscosity solution!

The Riemann Problem: Similarity Solutions

The quasi-linear PDE

$$\frac{\partial q}{\partial t} + \frac{\mathrm{d}f}{\mathrm{d}q}\frac{\partial q}{\partial x} = 0$$

for the Riemann problem

$$q(x,0) = \begin{cases} q_{\mathrm{l}} & \text{if } x < 0 \\ q_{\mathrm{r}} & \text{if } x > 0 \end{cases}$$

has similarity solutions of the form

$$q(x,t) = \tilde{q}\left(\frac{x}{t}\right) \quad .$$

Inserting this ansatz into the PDE gives for *t>0*

$$-\frac{x}{t^2}\,\tilde{q}' + \frac{\mathrm{d}f}{\mathrm{d}q}\,\frac{1}{t}\,\tilde{q}' = 0$$

with the solutions

$$\tilde{q}'\left(\frac{x}{t}\right) = 0 \;\Rightarrow\; \tilde{q}\left(\frac{x}{t}\right) = \mathrm{const} \qquad \text{or} \qquad \tilde{q}'\left(\frac{x}{t}\right) \neq 0 \;\Rightarrow\; \frac{\mathrm{d}f}{\mathrm{d}q}\left(\tilde{q}\left(\frac{x}{t}\right)\right) = \frac{x}{t}$$

$\rightarrow$ for Burgers' equation: *v=x/t*

Burgers' Equation: Expansion Waves

Smooth regions with $\partial v/\partial x > 0$ produce a rarefaction wave or expansion wave.

Steps with $v_{left} < v_{right}$:

- expansion shock with characteristics going out of it is a weak solution
- but: any perturbation or small (but non-zero) viscosity would smooth the step and cause a rarefaction wave (rarefaction fan)

Only solutions that fulfil an entropy condition are allowed.

Lax entropy condition: For a convex scalar conservation law, a discontinuity propagating with speed s satisfies the Lax entropy condition if $v_{left} > s > v_{right}$

→ expansion shocks are not allowed
an entropy condition destroys time-reversibility

Introduction to Numerical Hydrodynamics

3. Non-linear Advection

3.2 Numerical Examples for Burgers' Equation

Burgers' Equation: Numerical Schemes

conservation form

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = 0$$

quasi-linear form

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0$$

conservative and non-conservative upwind* schemes:

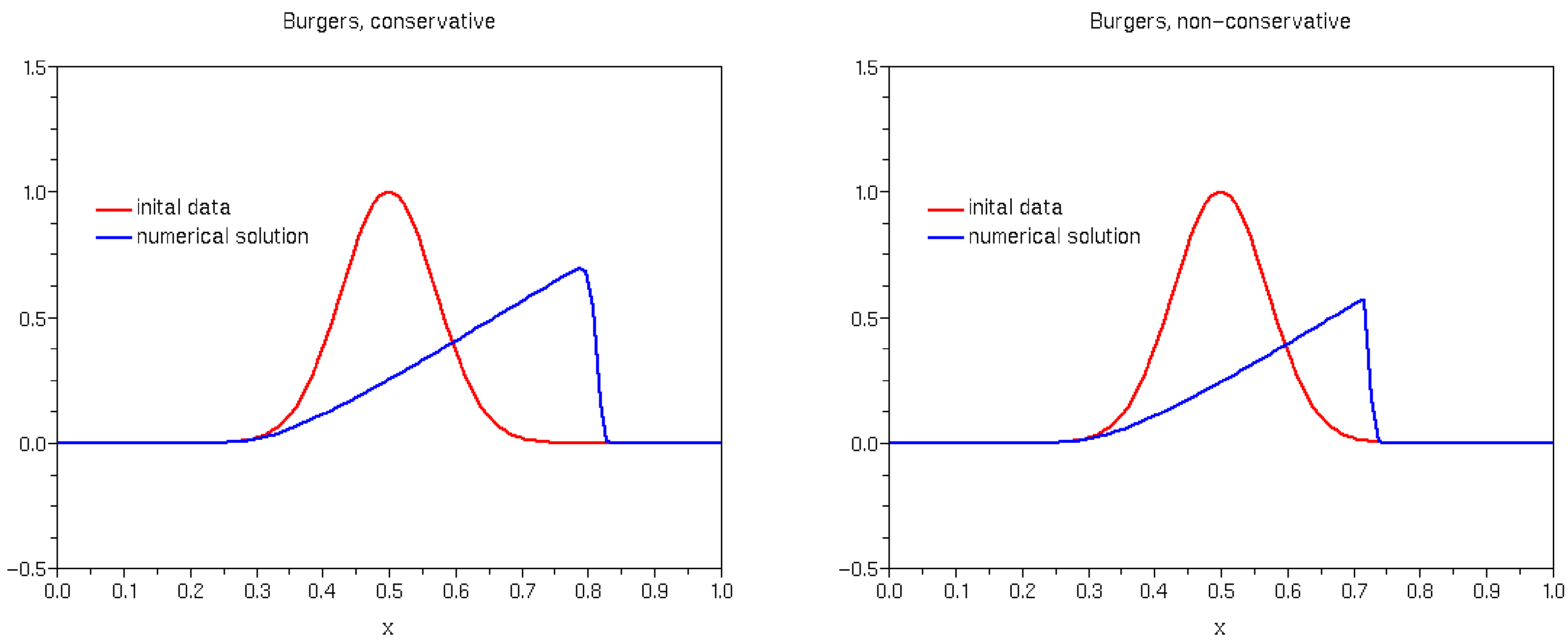
$$v_i^{n+1} = v_i^n - \frac{\Delta t}{\Delta x}\frac{1}{2}\left(\left(v_i^n\right)^2 - \left(v_{i-1}^n\right)^2\right)$$

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{\Delta x} v_i^n \left(v_i^n - v_{i-1}^n\right)$$

* for $v>0$

Burgers' Equation: Numerical Schemes

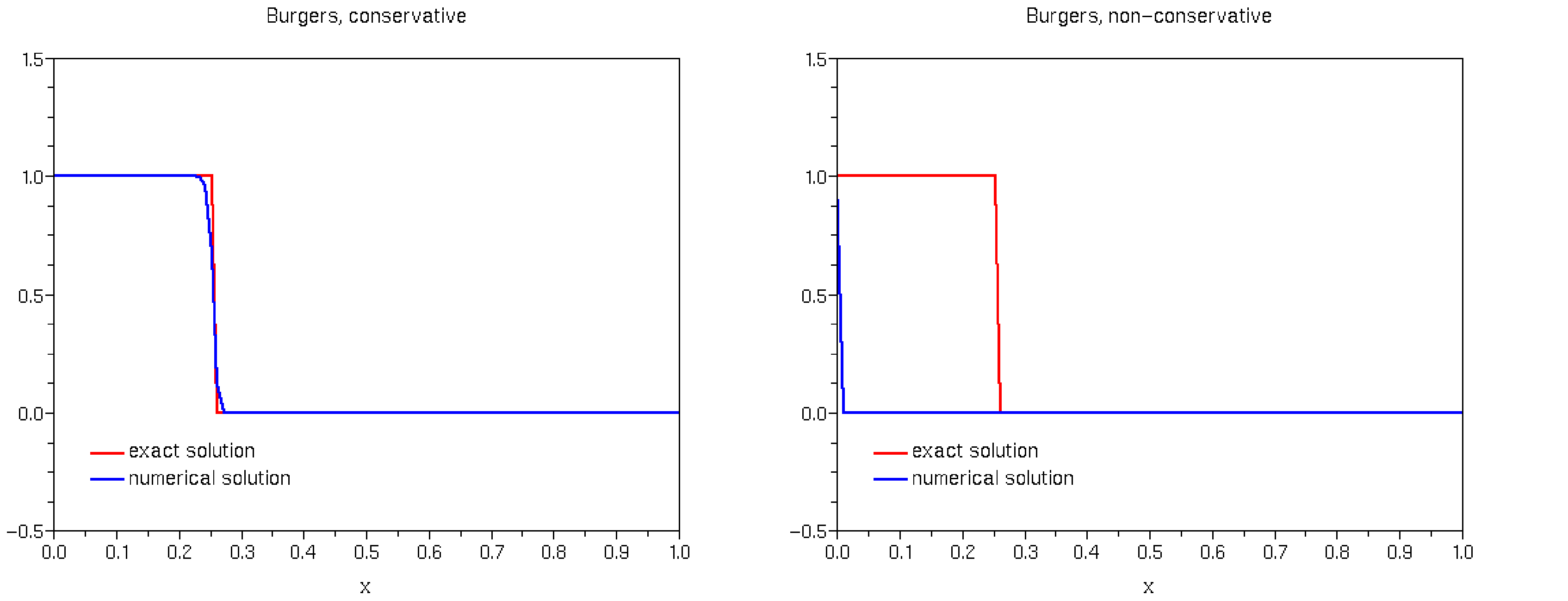
Example: Gaussian data



Initial data (red) and numerical solutions (blue) using the conservative (left panel) and non-conservative (right panel) upwind schemes. Note the change in the area under the curve and the wrong shock speed in the non-conservative case.

Burgers' Equation: Numerical Schemes

Example: Step function (Riemann, Case I: known shock speed)



Exact solution and numerical solution using the conservative (left panel) and non-conservative (right panel) upwind schemes. Clearly wrong shock speed (= 0) in the non-conservative case (right panel; no shock propagation, $v(x,t) = v(x,0)$ for all $t>0$) ...

Burgers' Equation: Numerical Schemes

conservation form

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2} v^2\right) = 0$$

quasi-linear form

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = 0$$

conservative and non-conservative
upwind* schemes:

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{\Delta x} \frac{1}{2}\left(\left(v_i^n\right)^2 - \left(v_{i-1}^n\right)^2\right)$$

$$\cancel{v_i^{n+1} = v_i^n - \frac{\Delta t}{\Delta x} v_i^n \left(v_i^n - v_{i-1}^n\right)}$$

* for *v>0*

wrong shock speed, area under v-curve not conserved

A conservative scheme is crucial !
... but is it sufficient?

Burgers' Equation: Numerical Schemes

conservation form

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = 0$$

conservative

upwind* scheme

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{\Delta x}\frac{1}{2}\left(\left(v_i^n\right)^2 - \left(v_{i-1}^n\right)^2\right)$$

* for $v>0$

Lax-Wendroff Theorem:

If the solution of a conservative method converges ($\Delta x \to 0$), it converges towards a weak solution of the conservation law.

Note: - convergence as such is not guaranteed
- selection of 'right' weak solution (entropy) not guaranteed

Burgers' Equation: Numerical Schemes

Flux splitting: adapting schemes written for one sign of the velocity to allow for both signs, still guaranteeing proper upwinding,

$$f(q) = f^+(q) + f^-(q) \quad \text{with} \quad \frac{\mathrm{d}f^+}{\mathrm{d}q} \geq 0 \ , \ \frac{\mathrm{d}f^-}{\mathrm{d}q} \leq 0 \ .$$

Example: extension of FTBS scheme, stable for both signs of v:

$$f^n_{i+\frac{1}{2}} = \begin{cases} f(\rho^n_i) & \text{if } v_{i+\frac{1}{2}} > 0 \\ f(\rho^n_{i+1}) & \text{if } v_{i+\frac{1}{2}} \leq 0 \end{cases} \qquad v_{i+\frac{1}{2}} = \begin{cases} \frac{f(q^n_{i+1}) - f(q^n_i)}{q^n_{i+1} - q^n_i} & \text{if } q^n_{i+1} \neq q^n_i \\ \frac{\mathrm{d}f}{\mathrm{d}q}(q^n_i) & \text{if } q^n_{i+1} = q^n_i \end{cases}$$

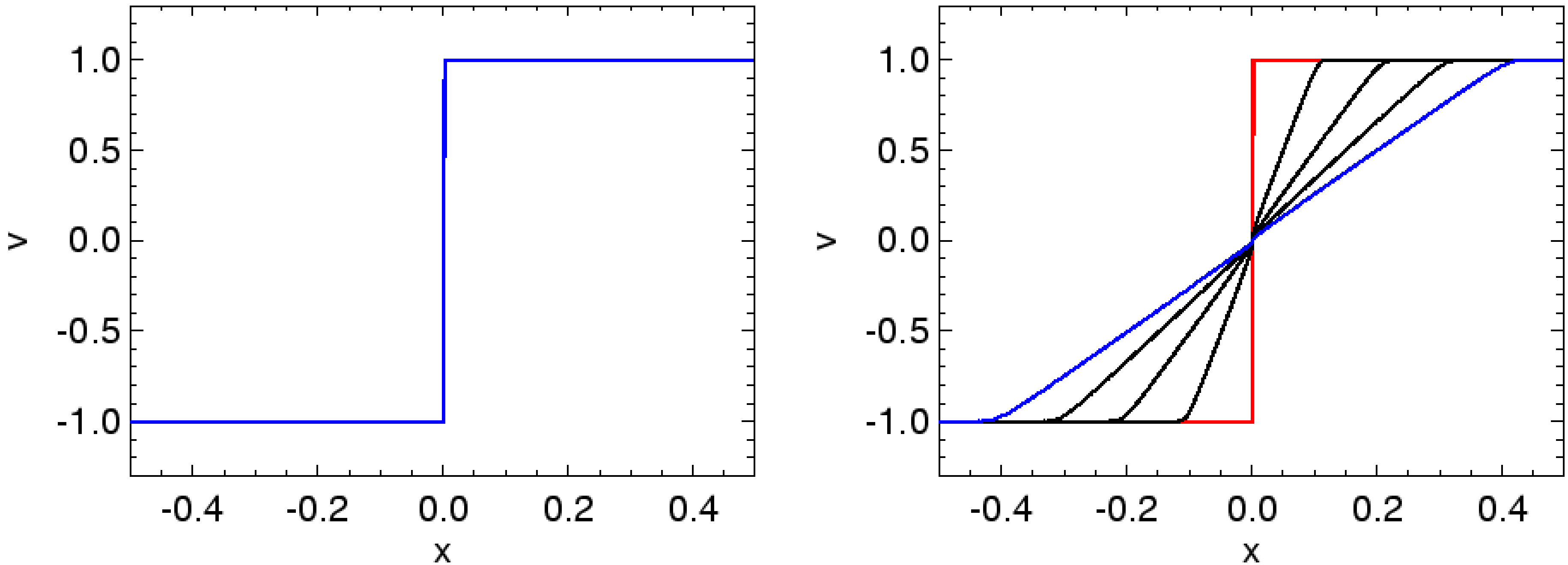
velocity at cell boundary

e.g., for Burgers' equation:

$$f^n_{i+\frac{1}{2}} = \begin{cases} \frac{1}{2}(v^n_i)^2 & \text{if } v_{i+\frac{1}{2}} > 0 \\ \frac{1}{2}(v^n_{i+1})^2 & \text{if } v_{i+\frac{1}{2}} \leq 0 \end{cases} \qquad v_{i+\frac{1}{2}} = \frac{1}{2}\left(v^n_i + v^n_{i+1}\right) \ .$$

Burgers' Equation: Numerical Schemes

Example: Expansion shock and rarefaction fan



Figures courtesy of Bernd Freytag

The initial data (left panel) should result in a rarefaction fan (right panel, black and blue curves). Instead, the extended FTBS scheme produces a stationary expansion shock (red curve) that violates the entropy condition (solution stationary because $f=1/2$ everywhere).

Burgers' Equation: Numerical Schemes

Entropy fix:

- for most Riemann problems, the extended FTBS scheme actually produces initially the right result
- only for a transonic rarefaction wave it differs (see last example)
- for Burgers' equation, the flux through the stagnation point ($v=0$) in a transonic rarefaction wave is $f = v^2/2 = 0$

$\Rightarrow$ extend the scheme with an additional branch (entropy fix):

$$f^n_{i+\frac{1}{2}} = \begin{cases} \frac{1}{2}\left(v^n_i\right)^2 & \text{if } v_{i+\frac{1}{2}} > 0 \text{ and } v_i \geq 0 \\ \frac{1}{2}\left(v^n_{i+1}\right)^2 & \text{if } v_{i+\frac{1}{2}} \leq 0 \text{ and } v_{i+1} \leq 0 \\ 0 & \text{if } v_i < 0 < v_{i+1} \end{cases} .$$

Alternative: entropy production by artificial viscosity, add an artificial diffusion term to the flux.

Burgers' Equation: Numerical Schemes

Concepts from the linear world:

- Consistency:
  the same (vanishing truncation error for Δt, $\Delta x \rightarrow 0$)
- Conservativity:
  the same (now even more important)
- Stability:
  - Linear stability: not directly applicable
  - Total variation diminishing (TVD): applicable
  - Alternative: base scheme on concepts that work in the linear case and perform lots of tests for the non-linear PDE

Burgers' Equation – Homework 3

Compute the numerical solution of Burgers equation

$$\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = 0$$

using the conservative and non-conservative upwind* schemes

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{\Delta x}\frac{1}{2}\left(\left(v_i^n\right)^2 - \left(v_{i-1}^n\right)^2\right)$$

$$v_i^{n+1} = v_i^n - \frac{\Delta t}{\Delta x} v_i^n \left(v_i^n - v_{i-1}^n\right)$$

* for $v>0$

(a) for the same set-up as in Homework 1&2 (initial conditions, range, numerical parameters, with v taking the role of ρ), and

(b) adding +1 to the initial data everywhere (to avoid $v=0$).

Compare the results to the exact solution (note: the discontinuity propagates with shock speed $s = (v_{left} + v_{right})/2$).

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