Introduction to Numerical Hydrodynamics and Radiative Transfer

Part II: Hydrodynamics, Lecture 7

HT 2012

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Introduction to Numerical Hydrodynamics

4. Non-Linear Hydrodynamics

4.1 New Challenges

Equations of Fluid Dynamics: Euler Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j + P \delta_{ij}) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \nabla \cdot \left((\frac{1}{2}\rho u^2 + \rho e + P) \boldsymbol{u} \right) = 0$$

describe the conservation of mass, momentum and energy, based on simplifying physical assumptions:

- no external forces (e.g. gravity, radiation pressure)
- no heating or cooling by radiation or heat conduction
- no viscosity (friction at microscopic level, shear)

Equations of Fluid Dynamics: Euler Equations

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mathematically speaking:

- a coupled system of (up to) 5 non-linear PDEs
- dependent on time and (up to) 3 space dimensions
- hyperbolic conservation laws

Equations of Fluid Dynamics: Euler Equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho u\right) = 0$$

$$\frac{\partial}{\partial t}\left(\rho u\right) + \frac{\partial}{\partial x}\left(\rho u u + P\right) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \frac{\partial}{\partial x}\left(\left(\frac{1}{2}\rho u^2 + \rho e + P\right)u\right) = 0$$

... in one space dimension, to simplify the following discussion

- a coupled system of 3 non-linear PDEs
- dependent on time and 1 space dimension
- hyperbolic conservation laws

The Euler Equations in Conservation Form

$$\frac{\partial}{\partial t}\,\boldsymbol{q} + \frac{\partial}{\partial x}\,\boldsymbol{F}(\boldsymbol{q}) = 0$$

$$\boldsymbol{q} = \begin{pmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho u^2 + \rho e \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} \rho u \\ \rho u u + P \\ (\frac{1}{2}\rho u^2 + \rho e + P) u \end{pmatrix}$$

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total energy:
kinetic + internal

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$$\rho e_{tot} = \rho e + \frac{1}{2}\rho u^2$$

$$\rho e_{tot} = \frac{P}{\gamma - 1} + \frac{1}{2}\rho u^2$$

equation of state
for polytropic gas:

$$\rho e = \frac{P}{\gamma - 1}$$

Mathematical Properties of PDEs

$$\frac{\partial}{\partial t}\,\boldsymbol{q} + \frac{\partial}{\partial x}\,\boldsymbol{F}(\boldsymbol{q}) = 0$$

is a hyperbolic system if the Jacobian matrix of the flux function

$$\boldsymbol{F}'(\boldsymbol{q}) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{q}}$$

has the following property: For each value of $\boldsymbol{q}$ the eigenvalues of $\boldsymbol{F}'(\boldsymbol{q})$ are real, and the matrix is diagonalizable, i.e., there is a complete set of linearly independent eigenvectors.

Mathematical Properties of PDEs

$$\frac{\partial}{\partial t} \boldsymbol{q} + \frac{\partial}{\partial x} \boldsymbol{F}(\boldsymbol{q}) = 0$$

conservative form

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quasilinear form

$$\frac{\partial}{\partial t} \boldsymbol{q} + \boldsymbol{F}'(\boldsymbol{q}) \frac{\partial}{\partial x} \boldsymbol{q} = 0$$

Mathematical Properties of PDEs

hyperbolic character of system $\Rightarrow$ important consequences for the propagation of information in the flow:

quantities called invariants are transported along characteristics

$$dx/dt = \lambda$$

where λ is an eigenvalue of the Jacobian matrix

$$\boldsymbol{F}'(\boldsymbol{q}) = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{q}}$$

invariant r is constant along the characteristic $\Rightarrow$

$$\frac{dr}{dt} = \frac{\partial r}{\partial t} + \frac{\partial r}{\partial x}\frac{dx}{dt} = 0 \qquad \text{or} \qquad \frac{\partial r}{\partial t} + \lambda \frac{\partial r}{\partial x} = 0$$

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example: advection equation
(constant velocity u)

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0$$

$x = u\,t$
$r = \rho$

The Euler Equations in Conservation Form

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equation of state
polytropic gas:

$$\rho e_{tot} = \frac{P}{\gamma - 1} + \frac{1}{2}\rho u^2$$

$$\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{q}} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2}(\gamma-3)u^2 & (3-\gamma)u & (\gamma-1) \\ \frac{1}{2}(\gamma-1)u^3 - u(\rho e_{tot}+P)/\rho & (\rho e_{tot}+P)/\rho - (\gamma-1)u^2 & \gamma u \end{pmatrix}$$

Mathematical Properties of PDEs

hyperbolic character of system $\Rightarrow$ important consequences for the propagation of information in the flow:

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Definition of upwind direction?

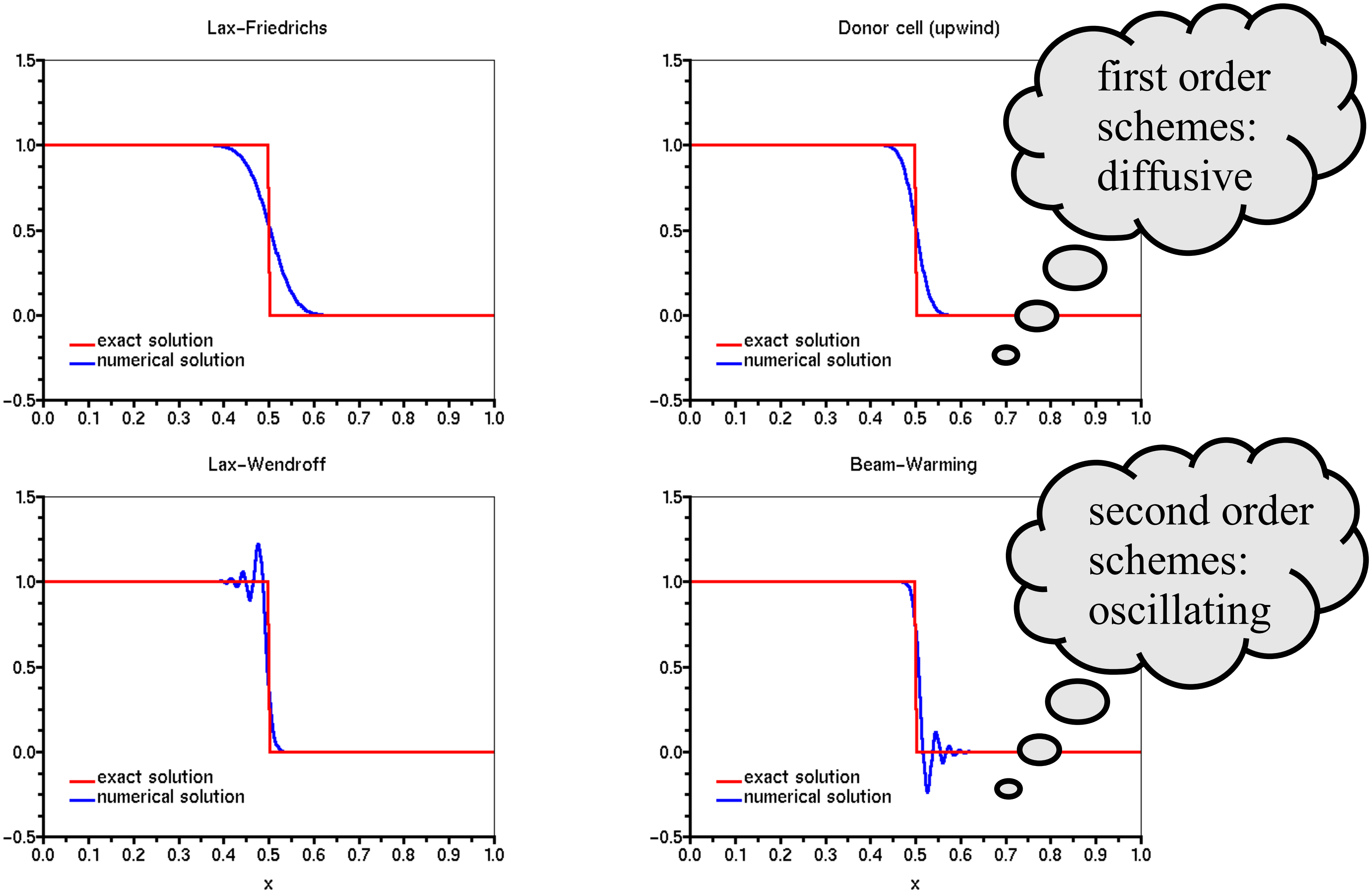
invariant r is constant along the characteristic $\Rightarrow$

$$\frac{dr}{dt} = \frac{\partial r}{\partial t} + \frac{\partial r}{\partial x}\frac{dx}{dt} = 0 \qquad \text{or} \qquad \frac{\partial r}{\partial t} + \lambda \frac{\partial r}{\partial x} = 0$$

example: Euler equations

eigenvalues λ $\Rightarrow$ velocities u, u+c, u–c (c = sound speed)

Homework 1 and 2 – Results



The Euler Equations: Numerical Challenges

Positivity of density:

To compute the velocity from momentum and density via $u = (\rho u)/\rho$ requires $\rho > 0$. An overshoot of the density to non-positive values is now disastrous.

Positivity of pressure:

The pressure $P(e,\rho)$ depends on the conserved quantities via

$$e = \frac{\rho e_{tot}}{\rho} - \frac{(\rho u)^2}{2\rho^2}$$

An overshoot in velocity may lead to negative pressure.

This restriction is so severe that in some cases the conservation of total energy might be given up in favour of a (non-conservative) formulation where the positivity of internal energy is guaranteed.

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4. Non-Linear Hydrodynamics

4.2 The Riemann Problem for the 1D Euler Equations

The Euler Equations in Conservation Form

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$$\boldsymbol{q} = \begin{pmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho u^2 + \rho e \end{pmatrix} \qquad \boldsymbol{F} = \begin{pmatrix} \rho u \\ \rho u u + P \\ (\frac{1}{2}\rho u^2 + \rho e + P) u \end{pmatrix}$$

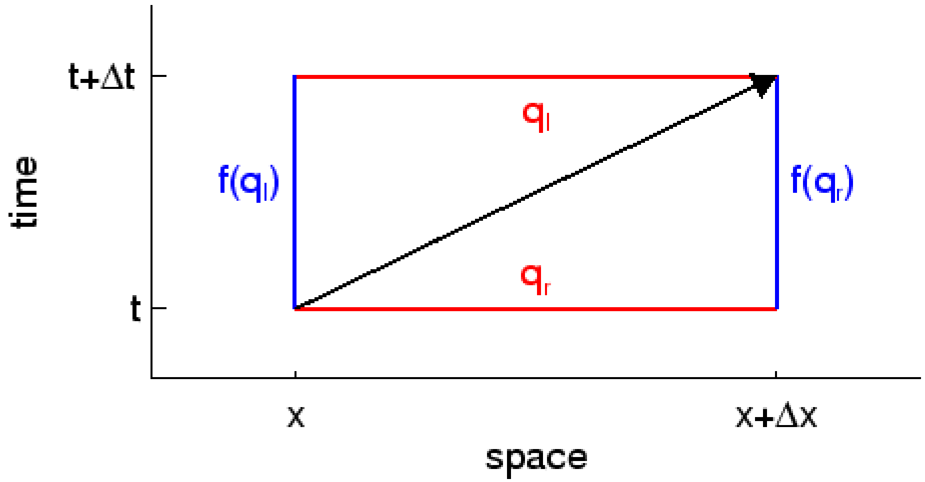
Riemann problem

- conservation law
- piecewise constant initial condition (discontinuity)

allows for analytical solution $\Rightarrow$ test case for numerical schemes!

The Riemann Problem: Shock Speed

In a time interval Δt
a shock with speed s travels
a distance $\Delta x = s\ \Delta t$.



Integration of the PDE

$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = 0$$

over Δt and Δx results in

$$\int_x^{x+\Delta x} [q(x, t+\Delta t) - q(x,t)]\, dx + \int_t^{t+\Delta t} [f(q(x+\Delta x, t)) - f(q(x,t))]\, dt = 0$$

For almost constant states and fluxes to the left and right we get

$$\Delta x\, q_{\mathrm{l}} - \Delta x\, q_{\mathrm{r}} + \Delta t\, f(q_{\mathrm{r}}) - \Delta t\, f(q_{\mathrm{l}}) = O\big(\Delta t^2\big) \quad .$$

For $\Delta x = s\ \Delta t$ and $\Delta t \to 0$ we get the shock speed

$$s = \frac{f(q_{\mathrm{r}}) - f(q_{\mathrm{l}})}{q_{\mathrm{r}} - q_{\mathrm{l}}}$$

Rankine-Hugoniot Conditions

The Rankine-Hugoniot jump conditions for the 1D Euler equations, describing a shock with speed s, become

$$s\ [\rho_r - \rho_l] = (\rho u)_r - (\rho u)_l$$
$$s\ [(\rho u)_r - (\rho u)_l] = (\rho u u + P)_r - (\rho u u + P)_l$$
$$s\ [(\rho e + \rho u^2/2)_r - (\rho e + \rho u^2/2)_l] = ((\rho e + \rho u^2/2 + P)u)_r - ((\rho e + \rho u^2/2 + P)u)_l$$

They can only be fulfilled for certain combinations of q_l and q_r .

An arbitrary Riemann problem typically causes more than one jump.

The Riemann Problem for the 1D Euler Equations

The state on each side is described by 3 values.

Each wave family can cause a discontinuity:

- sound waves ($u \pm c$) can cause shocks or rarefaction waves
- the material flow (u, entropy wave) can have a contact discontinuity

The solution of the Riemann problem (for convex – simple – equation of state) can comprise:

0 or 1 contact discontinuity
0, 1 or 2 shocks
0, 1 or 2 rarefaction waves
not more than 2 (shocks + rarefaction waves)

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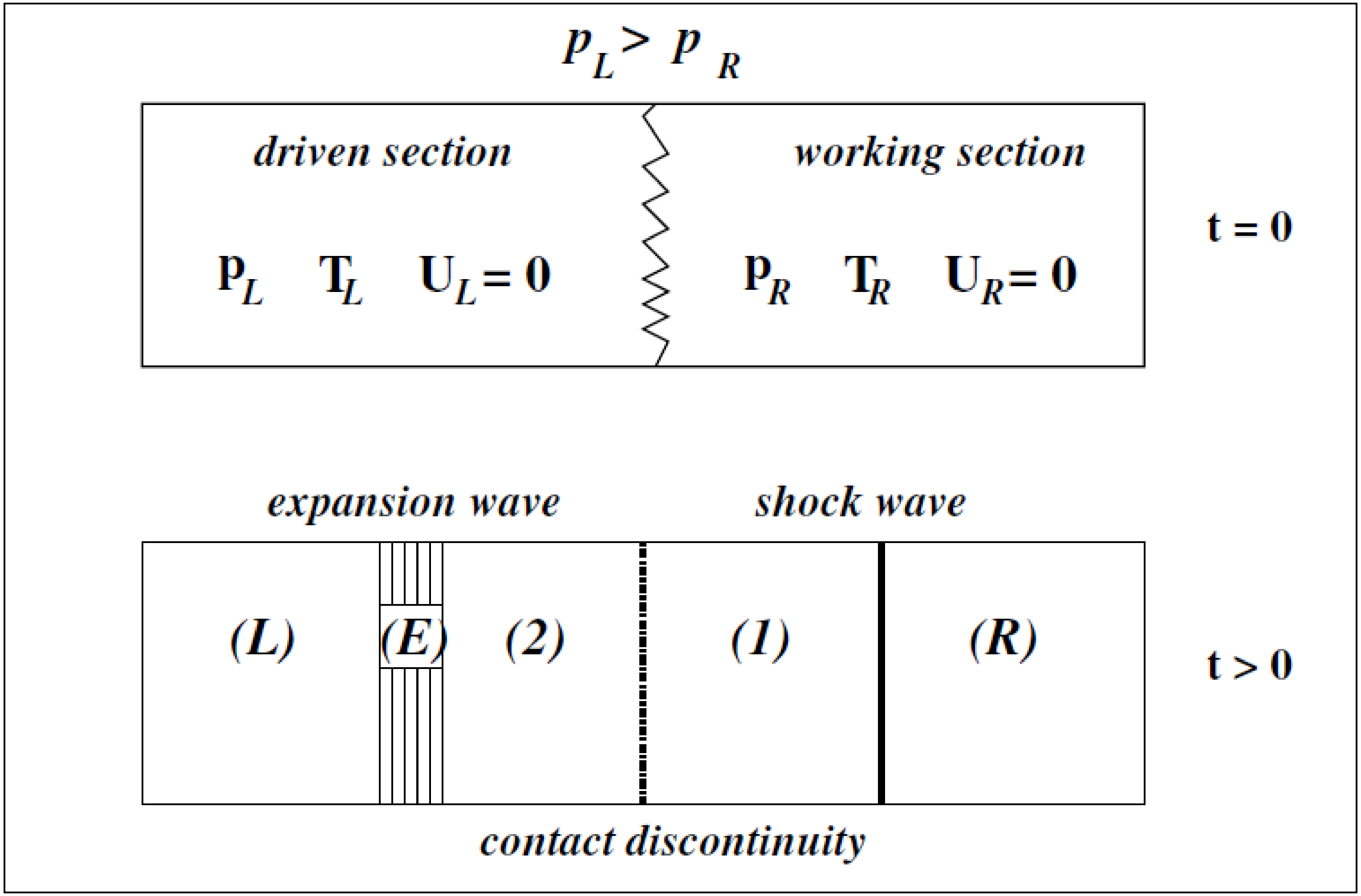
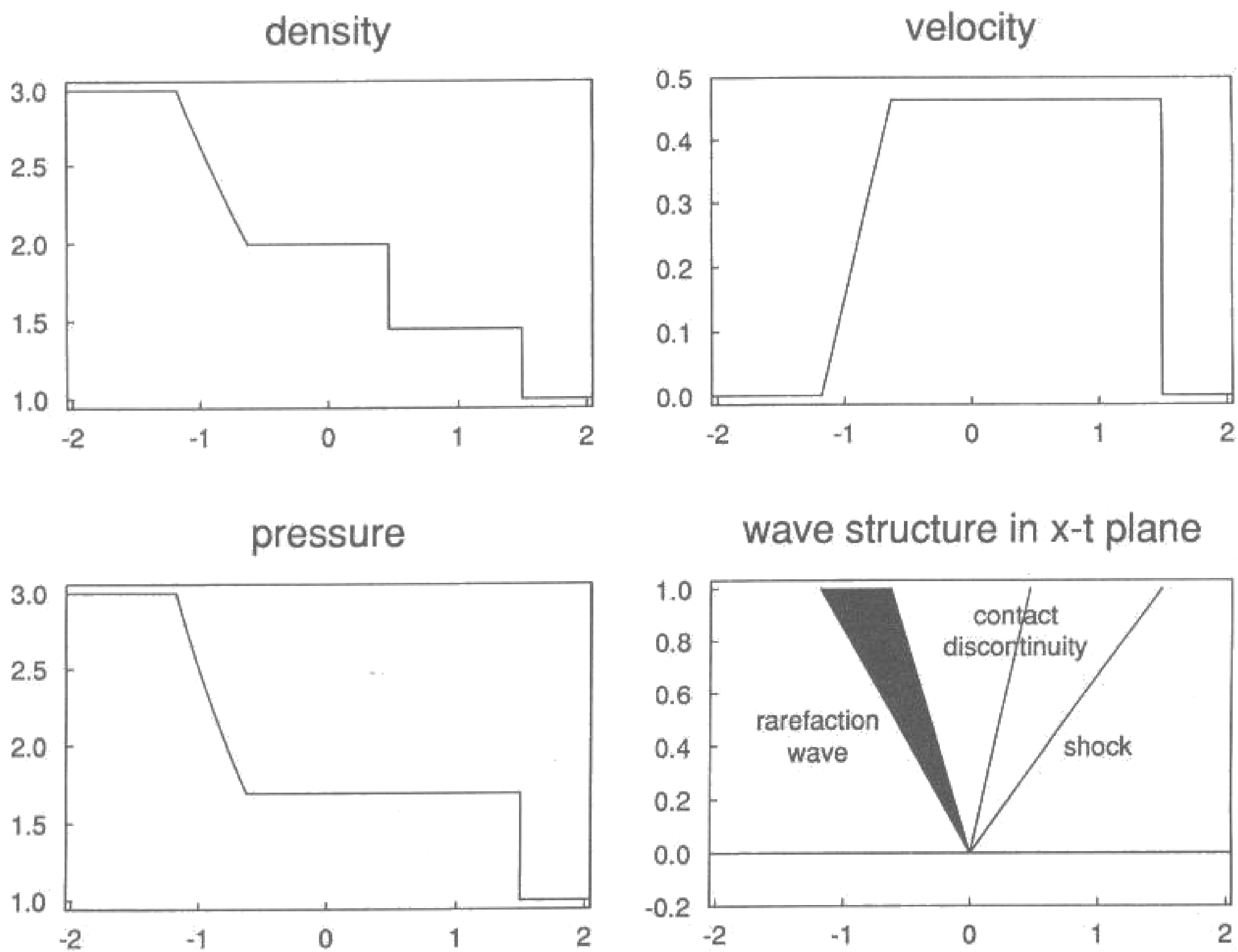
4.3 The Shock Tube Problem

Shock Tube Problem: Definition



Shock Tube Problem: Structure of the Solution



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Selected Literature

Selected Literature

General Background

LeVeque, R.J., *Numerical Methods for Conservation Laws,* Birkhäuser, 1990

Astrophysical Context

LeVeque, R.J., *Nonlinear Conservation Laws and Finite Volume Methods,* in: Computational Methods for Astrophysical Fluid Flow, Saas-Fee Advanced Course 27, eds. LeVeque R.J., Mihalas D., Dorfi E.A., Müller E., Springer, 1997

Flux Tube Problem

An Introduction to Scientific Computing: Twelve Computational Projects Solved with MATLAB, by I. Danaila, P. Joly, S.M. Kaber and M. Postel, Springer, 2007