Exercise 1: Linear Advection

- Solve the 1D linear advection equation

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} = 0$$

  for $u > 0$ and periodic boundary conditions in the spatial range $[-0.5, 0.5]$ with a set of representative schemes:

  - naive FTCS
  - donor cell (upwind)
  - Lax-Wendroff
  - Fromm
  - PLM: Minmod
  - PLM: van Leer
  - PLM: Superbee

- Test and compare the schemes (see below for details).
- Write a report containing some essential plots and a short evaluation of each scheme.

Detailed Questions:

- Naive scheme (FTCS): Check that this scheme is unstable for non-trivial initial conditions. Nevertheless, could the scheme be used in any way (e.g., for very smooth initial conditions, small time steps)?
- For the stable schemes: Use a Gaussian ($exp(-(x/0.1)^2)$) and a box-shaped function (1 for $abs(x) < 0.2$ and 0 elsewhere) as initial conditions. Choose a reasonable Courant number and keep it fixed. Simulate a complete cycle ($u\Delta t = 1$) and compare the numerical result with the exact result (= inital condition).
- Compare the results for different spatial resolutions (e.g., *25, 50, 100, 200* grid points).
- Measure the error with the 1-norm

$$N_1 = \frac{1}{i_{\mathrm{total}}} \sum_i \mathrm{abs}\left(\rho_i^{n_{\mathrm{total}}} - \rho_i^0\right) \quad ,$$

- the Euclidian 2-norm

$$N_2 = \frac{1}{i_{\mathrm{total}}} \left[\sum_i \mathrm{abs}\left(\rho_i^{n_{\mathrm{total}}} - \rho_i^0\right)^2 \right]^{1/2} \quad ,$$

- and the maximum-norm

$$N_{\max} = \max_i \mathrm{abs}\left(\rho_i^{n_{\mathrm{total}}} - \rho_i^0\right) \quad .$$

- How do the errors decrease with resolution (for the different schemes and initial conditions)?
- How many grid points are needed to preserve a structure (e.g., to push the N2 error below a certain limit?
- Lax-Wendroff and Fromm scheme: How much overshoot is acceptable? What density contrast in the initial condition can be allowed to be sure that the density remains positive everywhere?