# Gas dynamics

- **Assumptions**
  - particle mean free paths $\ll$ size of the region
    $\rightarrow$ volume elements with average velocity $u$, density $\rho$, pressure $P$, temperature $T$ $\rightarrow$ hydrodynamics equations: conservation of mass, momentum and energy
  - local equilibrium $\rightarrow$ Maxwell distribution for particle velocities within a volume element
  - plane parallel flow geometry (1D) $\rightarrow$ volume element d$V$ is a cuboid with length d$x$ along the flow and unit area perpendicular to flow
  - first order
- **Conservation equations**
  - **Mass conservation** (continuity equation)
    - mass of d$V$ = $\rho$ d$x$
    - no sources or sinks of material within d$V$

$$\begin{aligned}\frac{\partial}{\partial t}(\rho \mathrm{d}x) &= \overbrace{\rho u}^{\text{incoming}} - \overbrace{(\rho + \mathrm{d}\rho)(u + \mathrm{d}u)}^{\text{outgoing}} \\ &= -(\rho \mathrm{d}u + u \mathrm{d}\rho + \cancel{\mathrm{d}\rho \mathrm{d}u})\end{aligned}$$

$$\boxed{\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho \cdot u) = 0}$$

- mass loss and gain terms could be added
- time-independent: $\rho u$ = constant

– **Momentum conservation** (Euler's equation)

* momentum of $\mathrm{d}V = \rho \mathrm{d}x \cdot u$
* change of momentum due to fluid flow
  and gas pressure acting on the surface of $\mathrm{d}V$

$$\frac{\partial}{\partial t}(\rho u \mathrm{d}x) = \rho u^2 - \underbrace{(\rho + \mathrm{d}\rho)(u + \mathrm{d}u)^2} - \mathrm{d}P$$

$$\rho u^2 + 2\rho u \mathrm{d}u \cancel{+\rho \mathrm{d}u^2} + u^2 \mathrm{d}\rho \cancel{+2u\mathrm{d}\rho\mathrm{d}u + \mathrm{d}\rho\mathrm{d}u^2}$$

$$\rho\frac{\partial u}{\partial t} + u\,\overbrace{\left(-u\frac{\partial \rho}{\partial x} - \rho\frac{\partial u}{\partial x}\right)}^{\partial\rho/\partial t} + 2\rho u\frac{\partial u}{\partial x} + u^2\frac{\partial \rho}{\partial x} = -\frac{\partial P}{\partial x}$$

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}}$$

* further terms could be added,
  accounting for forces due to
  gravity,
  magnetic fields,
  radiation field,
  viscosity
* viscous force: due to "internal friction" in the fluid
  (resistivity of the fluid to the flow)

  $$\propto \frac{\partial^2 u}{\partial x^2}$$

  usually much smaller than force due to gas pressure,
  important in high-speed flows with large velocity gradients
* time-independent: $\rho u^2 + P = \text{constant}$

– **Energy conservation**

* thermodynamics – 1st law:
  heat change in a system
  = change in internal energy + work done on surroundings

  $$\mathrm{d}Q = \mathrm{d}U + P\mathrm{d}V$$

* internal energy for ideal gas:
  monoatomic: kinetic energy of translations
  $U = \frac{3}{2}kT$ per particle
  diatomic: + kinetic energy of rotations
  $U = \frac{5}{2}kT$
  polyatomic: $U = \frac{6}{2}kT$

* heat change given by specific heat capacities

  constant volume ($\mathrm{d}V = 0$): $(\frac{\mathrm{d}Q}{\mathrm{d}T})_V = (\frac{\mathrm{d}U}{\mathrm{d}T})_V \equiv Mc_V$

  $\longrightarrow \mathrm{d}Q = Mc_V\mathrm{d}T + Nk\mathrm{d}T$ (with ideal gas eos)

  constant pressure: $(\frac{\mathrm{d}Q}{\mathrm{d}T})_P \equiv Mc_P = Mc_V + Nk$

  $M$ …total mass, $N$ …total number of particles

* ratio of specific heat capacities

  $$\gamma \equiv \frac{c_P}{c_V} = \frac{c_V + Nk/M}{c_V}$$

  monoatomic gas : $c_V = \frac{1}{M}\frac{\mathrm{d}U}{\mathrm{d}T} = \frac{\mathrm{d}}{\mathrm{d}T}\frac{3}{2}kT\frac{N}{M} \rightarrow \boldsymbol{\gamma = \frac{5}{3}}$

  diatomic: $\gamma = \frac{7}{5}$

  polyatomic: $\gamma = \frac{4}{3}$

- **Energy conservation - limiting cases**
  - **adiabatic** flow – negligible heat transport:

    $$\mathrm{d}Q = 0$$

    $$\rightarrow P\mathrm{d}V = -Mc_V\mathrm{d}T$$

    use ideal gas law and thermodynamic relations

    $$\rightarrow \boxed{P = const. \cdot \rho^{\gamma}}$$

    example: Supernova explosion
  - **isothermal** flow – extremely efficient heat transport:

    heat transport timescale $\ll$ dynamic timescale

    $\rightarrow$ balance between heating and cooling

    $\rightarrow$ $\boxed{\text{constant temperature}}$

    ideal gas eos: $P = \rho \frac{kT}{\mu m_u}$

    $$\rightarrow \boxed{P = const. \cdot \rho}$$

    example: HII region
  - both: $P \propto \rho^n$ with $n = \gamma$ or $n = 1$

- **Perturbations in a gas**
  - Initial conditions: $P = P_0, \rho = \rho_0, u = u_0 = 0$
  - Small perturbations $P = P_0 + P_1, \rho = \rho_0 + \rho_1, u = u_1$
  - Conservation equations:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} \;=\; 0 \left[ -u_1 \frac{\partial \rho_1}{\partial x} - \rho_1 \frac{\partial u_1}{\partial x} \right]$$

$$\rho_0 \frac{\partial u_1}{\partial t} + \frac{\partial P_1}{\partial x} \;=\; \rho_0 \frac{\partial u_1}{\partial t} + a_0^2 \frac{\partial \rho_1}{\partial x} \;=\; 0 \left[ -\rho_1 \frac{\partial u_1}{\partial t} - u_1 \frac{\partial u_1}{\partial x} \right]$$

$$\boldsymbol{P_1} = P - P_0 \propto (\rho_0 + \rho_1)^n - \rho_0^n \propto n \rho_0^{n-1} \rho_1 = n \frac{P_0}{\rho_0} \rho_1 \equiv \boldsymbol{a_0^2 \rho_1} \left[ +O(\rho_1^2) \right]$$

$$\rightarrow \frac{\partial^2 \rho_1}{\partial t^2} - a_0^2 \frac{\partial^2 \rho_1}{\partial x^2} = 0$$

$\rightarrow$ **sound wave** (acoustic wave) with constant **sound speed** $\boldsymbol{a_0}$

  - Large perturbations
    sound speed depends on local $\rho$ and $P$: $a^2 = n \frac{P}{\rho}$
    isothermal flow: $a^2 \propto \frac{\rho}{\rho} = const.$
    adiabatic flow: $a \propto \frac{\rho^{\gamma/2}}{\rho^{1/2}} \propto \rho^{1/3}$ for $\gamma = \frac{5}{3}$

- Values of sound speed are of the same order as mean thermal velocities:

$$a = \sqrt{\frac{5}{3}\frac{kT}{\mu m_u}}, \langle v \rangle = \sqrt{\frac{8}{\pi}\frac{kT}{\mu m_u}}$$

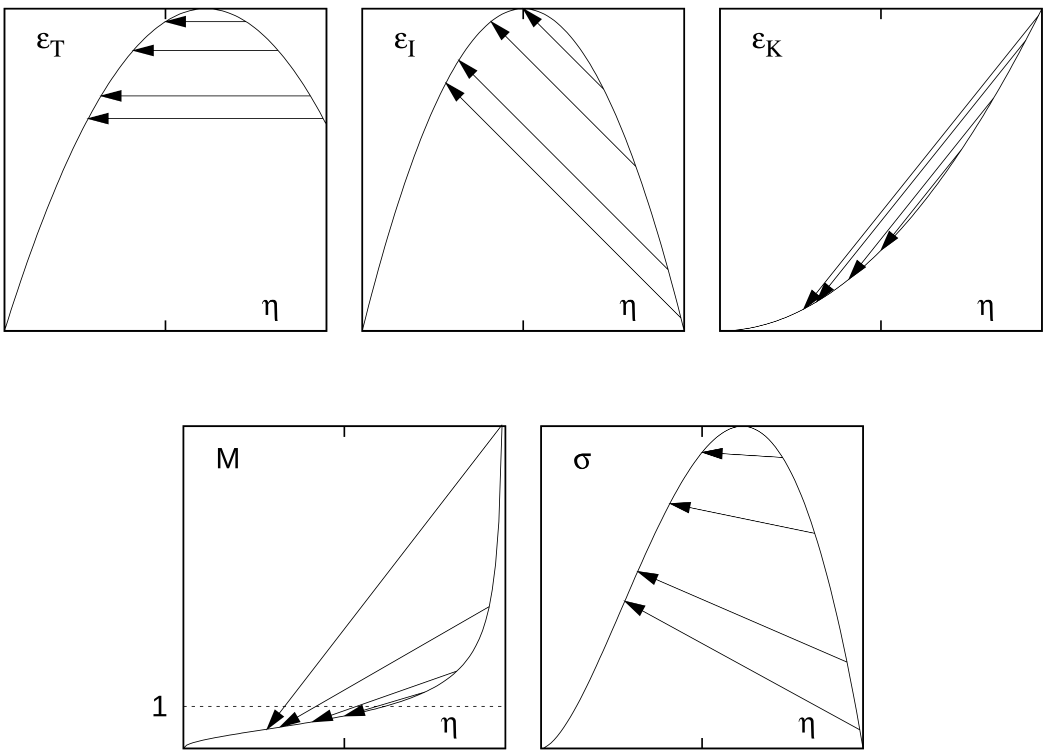
- Sound crossing time $t_{\text{cross}} = L/a$
  * time it takes for signal to cross a region of size $L$
  * compare $t_{\text{cross}}$ to evolutionary timescale
  * pressure gradient will be smoothed out within sound crossing time
  * changes occurring on timescales $< t_{\text{cross}}$ will survive
  * changes occurring on timescales $> t_{\text{cross}}$ will be damped

- Mach number $\equiv$ gas velocity/sound speed

$$M = u/a$$

- **Shock waves**
  - What are shock waves?
    see F.H. Shu, *The Physics of Astrophysics, Vol.II: Gas Dynamics*, University Science Books, 1992, Figures 15.2, 15.3, 15.4
  - Shocks in astrophysical situations
    * cloud-cloud collisions
    * HII regions expanding into neutral medium
    * stellar wind encountering medium
    * supernova blast waves
    * accretion onto stars
  - Shock jump conditions:
    mass, momentum, energy conservation across shock;
    also called "Rankine-Hugoniot" conditions
  - Analysis of jump conditions
    for dimensionless parameters
    * **conditions**:
      **1**: specific total energy $\varepsilon_T$ constant across shock
      **2**: entropy increases across shock
    * **results**:
      **1**: specific internal energy $\varepsilon_I$ increases across shock
      $\rightarrow$ temperature increases
      **2**: specific kinetic energy $\varepsilon_K$ decreases across shock
      $\rightarrow$ velocity decreases
      $\rightarrow$ density and pressure increase
      **3**: Mach number $> 1$ pre-shock and $< 1$ post-shock
      $\rightarrow$ only supersonic gas can produce a shock,
      gas is slowed down to subsonic by shock

# Jump conditions for dimensionless parameters



– Quantitative results for **strong shocks**:

∗ jump conditions:

$$\begin{aligned} \rho_0 u_0 &= \rho_1 u_1 \\ P_0 + \rho_0 u_0^2 &= P_1 + \rho_1 u_1^2 \\ \frac{u_0^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_0}{\rho_0} &= \frac{u_1^2}{2} + \frac{\gamma}{\gamma-1}\frac{P_1}{\rho_1} \end{aligned}$$

∗ in terms of upstream Mach number $M_0$

$$M_0^2 = \frac{u_0^2}{a_0^2} = \frac{\rho_0 u_0^2}{\gamma P_0} = \frac{\mu m_u u_0^2}{\gamma k T_0}$$

**1**: eliminate $p_1$ and $u_1$ from jump conditions

$$\frac{\rho_1}{\rho_0} = \frac{u_0}{u_1} = \frac{(\gamma+1)M_0^2}{(\gamma-1)M_0^2 + 2}$$

**2**: eliminate $\rho_1$ and $u_1$ from jump conditions

$$\frac{P_1}{P_0} = \frac{2\gamma M_0^2 - (\gamma-1)}{\gamma+1}$$

**3**: combine **1** and **2** with ideal gas law

$$\frac{T_1}{T_0} = \frac{[(\gamma-1)M_0^2 + 2][2\gamma M_0^2 - (\gamma-1)]}{(\gamma+1)^2 M_0^2}$$

∗ set $M_0 \gg 1$ and $\gamma = \frac{5}{3}$

$$\frac{\rho_1}{\rho_0} = \frac{u_0}{u_1} \approx \frac{\gamma+1}{\gamma-1} = 4$$

$$P_1 \approx \frac{2\gamma}{\gamma+1} M_0^2 P_0 = \frac{2}{\gamma+1}\rho_0 u_0^2 = \frac{3}{4}\rho_0 u_0^2$$

$$T_1 \approx \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} T_0 M_0^2 = \frac{2(\gamma-1)}{(\gamma+1)^2}\frac{\mu m_u}{k} u_0^2 = \frac{3}{16}\frac{\mu m_u}{k} u_0^2$$

– Strong shock moving in the rest frame

* change of reference frame
* shock velocity $V_s$
* upstream gas velocity $v_0$, downstream $v_1$
* transformation

$$\begin{aligned} u_0 &= v_0 - V_s \approx -V_s \quad (V_s \gg v_0) \\ u_1 &= v_1 - V_s \end{aligned}$$

* post-shock velocity

$$\frac{u_1}{u_0} = \frac{V_s - v_1}{V_s} = \frac{1}{4}$$

$$v_1 = \frac{3}{4} V_s$$

→ post-shock gas moves in the same direction as the shock
→ post-shock velocity is 3/4 of the shock velocity

* post-shock pressure

$$P_1 = \frac{3}{4} \rho_0 V_s^2$$

* post-shock temperature

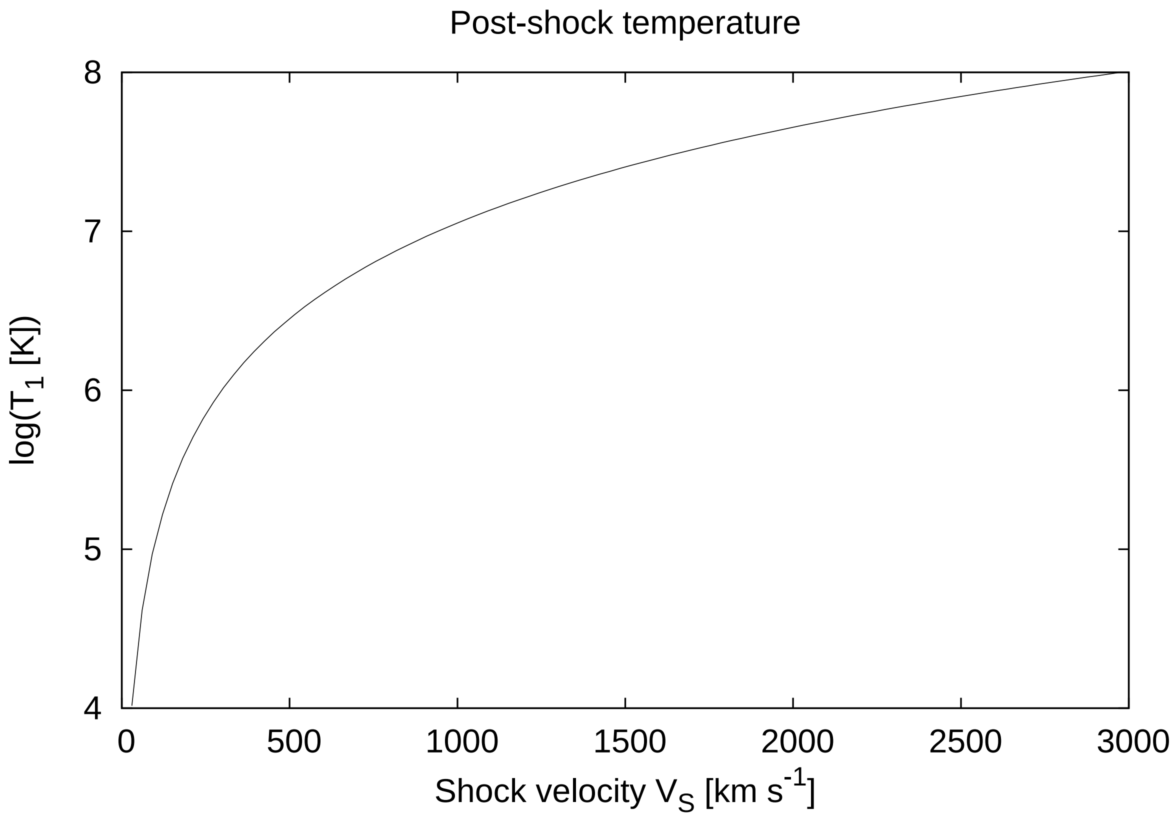
$$T_1 = \frac{3 \mu m_u}{16 k} V_s^2$$

* post-shock specific internal energy

$$e_{I,1} = \frac{3}{2} \frac{P_1}{\rho_1} = \frac{3}{2} \frac{3/4 \rho_0 V_s^2}{4 \rho_0} = \frac{9}{32} V_s^2$$

* post-shock specific kinetic energy

$$e_{K,1} = \frac{1}{2} v_1^2 = \frac{9}{32} V_s^2$$

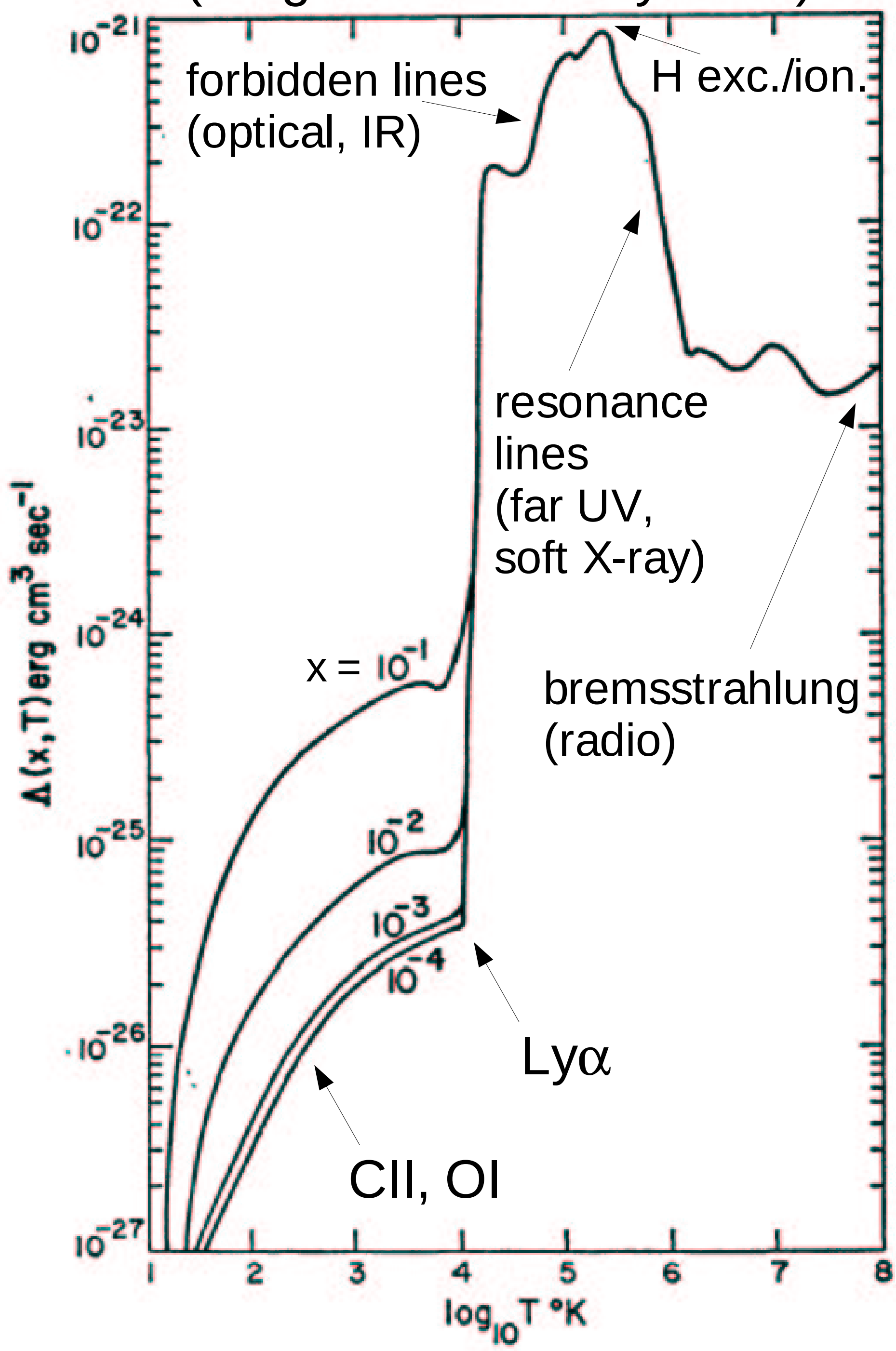

Post-shock temperature
8
7
6
5
4
log(T1 [K])
0
500
1000
1500
2000
2500
3000
Shock velocity VS [km s-1]

- **Cooling in shocked gas**
  - Large range of shock velocities
    $\rightarrow$ large range of post-shock temperatures
  - Cooling processes
    - collisional excitation or ionization
      followed by radiative de-excitation
    - bremsstrahlung at $\approx 10^8$ K
    - type of radiation depends on temperature
      $\rightarrow$ cooling function
    - between $10^5$ and $10^8$ K:
      cooling rate increases with decreasing temperature
      $\rightarrow$ gas at these temperatures is thermally unstable
  - Cooling time $t_c$:
    ratio between post-shock specific internal energy and cooling rate per unit mass
  - Cooling length $l_c$:
    distance travelled by gas during $t_c$
  - If cooling rate $\propto T^{-1/2} \rightarrow t_c \propto V_s^3$ and $l_c \propto V_s^4$
    $\rightarrow$ depend strongly on shock velocity
  - If $l_c \ll$ characteristic size of region
    $\rightarrow$ post-shock gas "immediately" cooled to pre-shock $T$
    $\rightarrow T = const. \rightarrow$ "isothermal" shock
    $\rightarrow$ with isothermal sound speed ($\gamma = 1$), for strong shock:

$$\frac{u_0}{u_1} = \frac{\rho_1}{\rho_0} = M_0^2$$

    $\rightarrow$ no limit for compression
    * in fixed reference frame: $v_1 \approx V_s$
    $\rightarrow$ post-shock gas moves with the same speed as the shock
    * post-shock pressure: $P_1 = \rho_0 V_s^2$

Interstellar cooling function
(Dalgarno & McCray 1972)




x ... fractional ionization